MATH 8, FALL 2025
ASSIGNMENT 2: PASCAL'S TRIANGLE

Pascal triangle is the following construction, where each number is obtained as the sum of two entries above it.

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

The k-th entry in n-th line is denoted by $\binom{n}{k}$. Note that both n and k are counted from 0, not from 1: for example, $\binom{2}{1} = 2$. Thus, these numbers are defined by these rules:

$$
\begin{aligned}
&\binom{n}{0} = \binom{n}{n} = 1 \\
&\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{for } 1 \le k \le n-1
\end{aligned}
\tag{1}
$$

It turns out that these are the same numbers as the number of ways to choose k items out of n:

$$\binom{n}{k} = {}_nC_k = \text{number of ways to choose } k \text{ items out of } n$$

Indeed, it suffices to prove that ${}_nC_k$ satisfy the same relations (1).

Suppose I have a collection of n books, and from these I must choose k books to place on my coffee table; by definition, there are ${}_nC_k$ ways to do that.

Suppose one of my n books is H : "The Hobbit". Then one can count all ways to choose k books as follows:

1. We can include "The Hobbit" in our choice. Then we also need to choose $k-1$ other books from the remaining $n-1$; thus, there are ${}_{n-1}C_{k-1}$ such choices.
2. we can decide **not** to include "The Hobbit". This leaves us with $n-1$ books to choose from, so there are ${}_{n-1}C_k$ such choices.

Therefore, we get the formula

$${}_nC_k = {}_{n-1}C_{k-1} + {}_{n-1}C_k$$

so ${}_nC_k = \binom{n}{k}$.

This gives one way to prove the explicit formula for these numbers:

$$\binom{n}{k} = {}_nC_k = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} \tag{2}$$

(see problem 5).

PROBLEMS

General note: In this homework assignment (and in all other assignments in this class), many problems are non-trivial and require some thought. Try to start early. You are not expected to be able to solve all of the problems, so do not be discouraged if you can't solve some of them. Please make sure that you show not just the answer but also the solution, i.e. your reasoning showing how you arrived to this answer. Ideally, your solution should be such that someone who doesn't know how to solve this problem can read it and follow your arguments.

It is enough if you can write the answers in terms of factorials and binomial coefficients — it is not necessary to actually compute them: an answer like $13!$ or $\binom{10}{5}$ is good enough.

1. How many ways are there to choose a committee of 5 people from a class of 32?

2. 5 kids come to a store to choose Halloween costumes. The store sells 25 different costumes. Assuming the store has enough stock for the kids to choose the same costume if they want, in how many ways can the kids choose the costumes? What if they want to choose so that all costumes are different?

3. Suppose I flip a coin 4 times, and I record the result each time (for example, the coin may land heads then tails then heads, then again heads which I will write as $HTHH$, where order matters). I will refer to this four letter combination as the final result - for example, $HHHH$ is the only final result that has no tails.
   (a) How many final results are there with exactly one tail?
   (b) How many final results are there with exactly two tails?
   (c) If I now toss the same coin 20 times, what is the probability that exactly half will be tails?

4. Five octopuses are working at the beach's local landfood restaurant. They want to assign lunch shifts, so that some of the octopuses can have lunch from 12:00pm to 1:00pm, and the others can have lunch from 1:00pm to 2:00pm.
   (a) If they decide to have two octopuses take the first shift and three take the second, how many possible ways are there to assign shifts?
   (b) If the noontime hour is especially busy and they decide to have just one octopus take the first shift and the remaining four take the second shift, how many ways are there to assign shifts?

5. Verify that if we define numbers $\binom{n}{k}$ by formula (2), then so defined numbers satisfy relations (1). [Hint: if it makes it easier, can you first do the argument for $n = 17$, $k = 5$]

6. Are there any rows in the Pascal triangle where all numbers are odd? Which rows are they? Can you prove your answer?

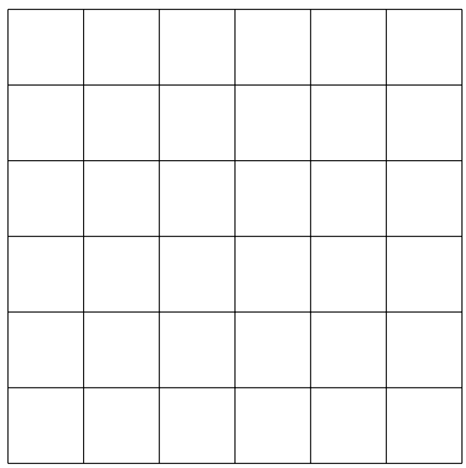
7. What is the sum of all numbers in n-th row of Pascal triangle? Can you guess the pattern — and once you guessed it, justify your guess.

8. How many ways are there to place two rooks on a chessboard so that they are not attacking each other? [For those of you who are unfamiliar, a chessboard is an 8×8 grid of squares, and *rooks* are pieces that can occupy any one of these individual squares, and may attack any other piece that's in the same row or column of the board as itself.]

9. How many different paths are there on 6×6 chessboard connecting the lower left corner with the upper right corner such that

- The path always goes to the right or up, never to the left or down.
- The path never goes below the diagonal (being on diagonal is OK)

Hint: start filling the chessboard with numbers, putting in each cell the number of paths leading to it. Start at lower left corner.



***10.** Suppose I flip a coin 12 times, and record the sequence of results (for example, HTHTHTHHTTHH). How many sequences of coin tosses are there for which no two consecutive tosses are heads (i.e., HH does not appear)? [Hint: let $f(n)$ be the number of such sequences when there are n coin tosses, and write a recursion for $f(n)$.]