MATH 8
ASSIGNMENT 16: EUCLIDEAN GEOMETRY 5: RULER AND COMPASS CONSTRUCTIONS
FEB 8, 2026

1. Axioms

Axiom 1. *For any two distinct points A, B, there is exactly one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.*

Axiom 2. *If distinct points A, B, C are on the same line, exactly one is between the other two; if point B is between A and C, then $AC = AB + BC$.*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*

Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure to the right (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*
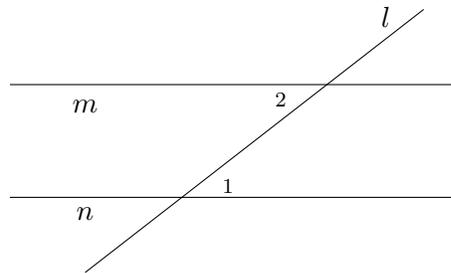
Axiom 5 (SAS Congruence). *If triangles $\triangle ABC$ and $\triangle A'B'C'$ have two congruent sides and a congruent angle between these sides, then the triangles are congruent: if $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\angle ABC \cong \angle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 6 (ASA Congruence). *If two triangles have two congruent angles and a congruent side between these angles, then the triangles are congruent.*

Axiom 7 (SSS Congruence). *If two triangles have three sides congruent, then the triangles are congruent.*

2. Theorems summary

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be vertical angles. Then $m\angle 1 = m\angle 3$.*

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Theorem 8. *Sum of angles of a triangle is equal to* $180°$.

Theorem 9 (Base angles equal)**.** *If* $\triangle ABC$ *is isosceles, with base* AC, *then* $m\angle A = m\angle C$.
  Conversely, if $\triangle ABC$ *has* $m\angle A = m\angle C$, *then it is isosceles, with base* AC.

Theorem 10. *If* B *is the apex of the isosceles triangle* ABC, *and* BM *is the median, then* BM *is also the altitude, and is also the angle bisector, from* B.

Theorem 11. *In* $\triangle ABC$, *if* $m\angle A > m\angle C$, *then we must have* $BC > AB$.

Corollary. *Let* P *be a point not on line* l, *and let* $Q \in l$ *be such that* $PQ \perp l$. *Then for any other point* R *on line* l, *we have* $PR > PQ$, *i.e. the perpendicular is the shortest distance from a point to a line.*

Theorem 12. *In* $\triangle ABC$, *if* $BC > AB$, *then we must have* $m\angle A > m\angle C$.

Theorem 13 (The triangle inequality)**.** *In* $\triangle ABC$, *we have* $AB + BC > AC$.

Theorem 14. *Let* $ABCD$ *be a parallelogram. Then*
  - $AB = DC$, $AD = BC$
  - $m\angle A = m\angle C$, $m\angle B = m\angle D$
  - *The intersection point* M *of diagonals* AC *and* BD *bisects each of them.*

Theorem 15. *Any quadrilateral* $ABCD$ *is a parallelogram if any one of the following conditions is true. In this case, all other conditions are also true.*
  - *its opposite sides are equal (*$AB = CD$ *and* $AD = BC$*),* **OR**
  - *two opposite sides are equal and parallel (*$AB = CD$ *and* $AB \parallel CD$*),* **OR**
  - *its diagonals bisect each other (*$AM = CM$ *and* $BM = DM$, *where* $AC \cap BD = M$*),* **OR**
  - *its opposing angles are equal (*$\angle BAD = \angle BCD$ *and* $\angle ABC = \angle ADC$*).*

Theorem 16. *Let* $ABCD$ *be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

Theorem 17. *If* DE *is the midline of* $\triangle ABC$, *then* $DE = \frac{1}{2}AC$, *and* $\overline{DE} \parallel \overline{AC}$.

Theorem 18. *Let* $ABCD$ *be a trapezoid, with bases* AD *and* BC, *and let* EF *be the midline (i.e.* E, F *be midpoints of sides* AB, CD *respectively).*
  Then $\overline{EF} \parallel \overline{AB}$, *and* $EF = (AD + BC)/2$.

Theorem 19. *Let* $ABCD$ *be a rectangle. Then it is a parallelogram; in particular, the opposite sides are equal. Moreover, the diagonals are equal.*

Theorem 20. *Given two distinct points* A, B, *a point* P *is equidistant from them (i.e.* $AP = BP$*) if and only if* P *lies on the perpendicular bisector of* AB.

Theorem 21. *Given an angle* $\angle BAC$ *and point* P *inside this angle,* P *is equidistant from the two sides of the angle if and only if* P *lies on the angle bisector of* $\angle BAC$.

Theorem 22. *In a triangle* $\triangle ABC$,
  1. *The three angle bisectors intersect at a single point inside the triangle; this point is called the* incenter *and is at equal distance from all three sides*
  2. *The three perpendicular bisectors to the sides intersect at a single point; this point is called the* circumcenter *and is at equal distance from all three vertices.*
  3. *The lines containing the three altitudes of the triangle intersect at a single point; this point is called the* orthocenter *of the triangle*
  4. *The three medians intersect at a single point inside the triangle; this point is called the* centroid, *or center of mass, of the triangle. It divides each of the medians in the proportion* $2 : 1$.

Theorem 23. *If AB is a chord of a circle, then the center O of this circle lies on the perpendicular bisector of AB.*

Theorem 24. *Let C be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular OP from O to l. Then:*
- *If $d > r$, then C and l do not intersect.*
- *If $d = r$, then C intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is tangent to C at P.*
- *If $d < r$, then C intersects l at two distinct points.*

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

Theorem 25. *Let C be a circle with center O, and let l be a line through a point A on C. Then l is tangent to C if and only if $l \perp \overleftrightarrow{OA}$.*
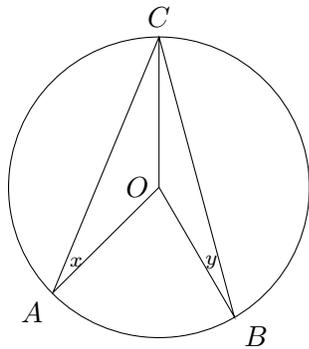
Theorem 26.
1. *Given a triangle $\triangle ABC$, there is a unique circle that contains all three vertices of the triangle. This is called the circumscribed circle; its center is the intersection point of the three perpendicular bisectors to the sides of the triangle.*
2. *Given a triangle $\triangle ABC$, there is a unique circle that is tangent to all three sides of the triangle. This is called the inscribed circle; its center is the intersection point of the three angle bisectors.*

Theorem 27. *Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B. Then $\overline{AB} \perp \overline{O_1O_2}$.*
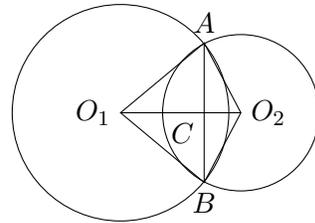
Theorem 28. *Let C_1, C_2 be circles that are both tangent to line m at point A. Then these two circles have only one common point, A. Such circles are called* tangent.

Theorem 29. *Let A, B, C be on a circle S with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in S. Conversely, given points A, B on S and a point C such that $\angle ACB = \frac{1}{2}\angle AOB$, C must lie on the circle S.*

Corollary. *Point C is on a circle with diameter AB if and only if $m\angle ACB = 90°$.*

Theorem 30. *Consider a circle S and an angle whose vertex C is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\widehat{AB}$ and $\widehat{A'B'}$. Then $m\angle C = \frac{1}{2}(\widehat{AB} - \widehat{A'B'})$.*

Similarly, if point C is inside the circle, then $m\angle C = \frac{1}{2}(\widehat{AB} + \widehat{A'B'})$.
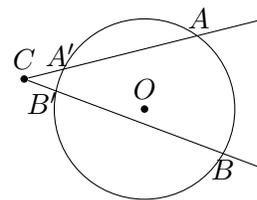
Many problems in geometry are about constructing various geometric objects. Traditionally such constructions are done using ruler and compass (or better, straightedge and compass). This means that we allow ourselves the following operations:

- Draw a line through two previously constructed points.
- Draw a circle with a given center and passing through another given point.
- Mark intersection points of already constructed lines or circles so that they can be used for subsequent constructions

Note that we are not allowed to measure distances and do arithmetic operations, or things like "start moving this ruler until we hit that point".

As an example, consider the following questions.

Problem. Find the midpoint of given segment AB.

Construction. Draw a circle centered at A and with radius AB; similarly, draw a circle with center B and radius BA. Let C, C' be the intersection points of these circles. Then $ACBC'$ is a rhombus, so its diagonals are perpendicular and bisect each other. Thus, CC' is perpendicualr bisector of AB, so the intersection point of CC' and AB is the midpoint of AB.
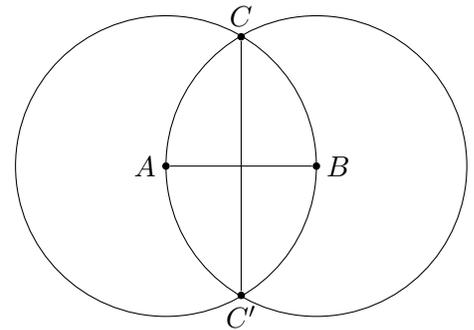
Problem. Given a circle S with center at point O, and a point A outside of this circle, construct a tangent line to S from A

Construction. Note that if B is the tangency point for such a line, then AB is perpendicualr to the radius OB. Thus, by Corollary to theorem 29, B must be on the circle with diameter AO.

This suggests this construction:

1. Find midpoint M of AO (same as in the previous problem)
2. Draw circle with cengter at M and radius AM
3. Let B be one of the intersection points of this circle and original circle S; draw line AB.
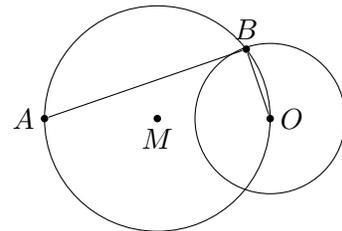
Note that there are two possible solutions: these two circles intersect at two points.

A lot of problems like this are in the app *Euclidea* (also available online at *euclidea.xyz*).

HOMEWORK

1. Show that if AB, AB' are two tnagent lines from A to the same circle S, then they have the same length: $AB = AB'$.

2. Prove the RHS congruence test: if $\triangle ABC$, $\triangle A'B'C'$ are right triangles: $m\angle C = 90°$, $m\angle C' = 90°$, and $AB = A'B'$, $AC = A'C'$, the these triangles are congruent. [Hint: construct circles with diameters AB, $A'B'$ and show that triangles $\triangle AOC$, $\triangle A'O'C'$ are congruent.]

3. Do levels Alpha to Iota in Euclidea. [As a proof, show the picture with unlocked level Kappa]