MATH 8
ASSIGNMENT 16: EUCLIDEAN GEOMETRY 4: INSCRIBED ANGLE THEOREM
FEB 1, 2026

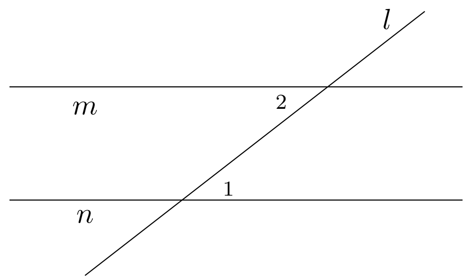
1. Axioms

Axiom 1. *For any two distinct points A, B, there is exactly one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.*

Axiom 2. *If distinct points A, B, C are on the same line, exactly one is* between the other two*; if point B is between A and C, then $AC = AB + BC$.*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*

Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure to the right (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle A'B'C'$ have two congruent sides and a congruent angle between these sides, then the triangles are congruent: if $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\angle ABC \cong \angle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a congruent side between these angles, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

2. Theorems summary

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be vertical angles. Then $m\angle 1 = m\angle 3$.*

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Theorem 8. *Sum of angles of a triangle is equal to* $180°$.

Theorem 9 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

Theorem 10. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*

Theorem 11. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Corollary. *Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$. Then for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.*

Theorem 12. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

Theorem 13 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

Theorem 14. *Let $ABCD$ be a parallelogram. Then*
- $AB = DC,\ AD = BC$
- $m\angle A = m\angle C,\ m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

Theorem 15. *Any quadrilateral $ABCD$ is a parallelogram if any one of the following conditions is true. In this case, all other conditions are also true.*
- *its opposite sides are equal ($AB = CD$ and $AD = BC$), **OR***
- *two opposite sides are equal and parallel ($AB = CD$ and $AB \parallel CD$), **OR***
- *its diagonals bisect each other ($AM = CM$ and $BM = DM$, where $AC \cap BD = M$), **OR***
- *its opposing angles are equal ($\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$).*

Theorem 16. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

Theorem 17. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

Theorem 18. *Let $ABCD$ be a trapezoid, with bases AD and BC, and let EF be the midline (i.e. E, F be midpoints of sides AB, CD respectively).*
Then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.

Theorem 19. *Let $ABCD$ be a rectangle. Then it is a parallelogram; in particular, the opposite sides are equal. Moreover, the diagonals are equal.*

Theorem 20. *Given two distinct points A, B, a point P is equidistant from them (i.e. $AP = BP$) if and only if P lies on the perpendicular bisector of AB.*

Theorem 21. *Given an angle $\angle BAC$ and point P inside this angle, P is equidistant from the two sides of the angle if and only if P lies on the angle bisector of $\angle BAC$.*

Theorem 22. *In a triangle $\triangle ABC$,*
1. *The three angle bisectors intersect at a single point inside the triangle; this point is called the* incenter *and is at equal distance from all three sides*
2. *The three perpendicular bisectors to the sides intersect at a single point; this point is called the* circumcenter *and is at equal distance from all three vertices.*
3. *The lines containing the three altitudes of the triangle intersect at a single point; this point is called the* orthocenter *of the triangle*
4. *The three medians intersect at a single point inside the triangle; this point is called the* centroid, *or center of mass, of the triangle. It divides each of the medians in the proportion $2:1$.*

Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $OP = r$.

Given a circle with center O,

- A radius is a line segment from O to a point A on the circle
- A chord is a line segment between distinct points A, B on the circle
- A diameter is a chord that passes through O,

Theorem 23. *If AB is a chord of a circle, then the center O of this circle lies on the perpendicular bisector of AB.*

Theorem 24. *Let C be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular OP from O to l. Then:*

- *If $d > r$, then C and l do not intersect.*
- *If $d = r$, then C intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is tangent to C at P.*
- *If $d < r$, then C intersects l at two distinct points.*

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

Theorem 25. *Let C be a circle with center O, and let l be a line through a point A on C. Then l is tangent to C if and only if $l \perp \overleftrightarrow{OA}$.*

Proof. By definition, if l is the tangent line to C, then it has only one common point with C, and this point is the base of the perpendicular from O to l; thus, OA is the perpendicular to l.

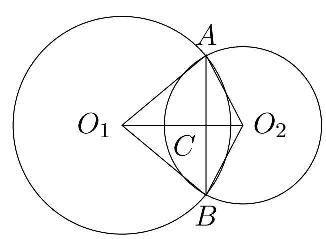
Conversely, if $OA \perp l$, it means that the distance from l to O is equal to the radius (both are given by OA), so l is tangent to C. □

Theorem 26.

1. *Given a triangle $\triangle ABC$, there is a unique circle that contains all three vertices of the triangle. This is called the circumscribed circle; its center is the intersection point of the three perpendicular bisectors to the sides of the triangle.*
2. *Given a triangle $\triangle ABC$, there is a unique circle that is tangent to all three sides of the triangle. This is called the inscribed circle; its center is the intersection point of the three angle bisectors.*

RELATIVE POSITION OF TWO CIRCLES

Theorem 27. *Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B. Then $\overline{AB} \perp \overline{O_1O_2}$.*



Proof. Let l be the perpendicular bisector of AB. By Theorem 23, l contains both centers: $O_1 \in l$, $O_2 \in l$. Thus, $l = \overline{O_1O_2}$, so $\overline{O_1O_2}$ is the perpendicaulr bisector of AB; in particular, they are perpendicular. □

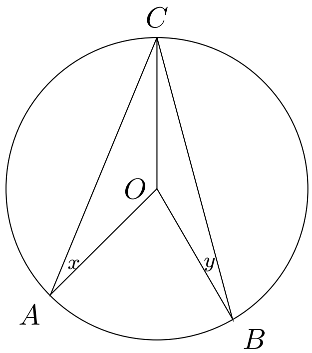
Theorem 28. *Let C_1, C_2 be circles that are both tangent to line m at point A. Then these two circles have only one common point, A. Such circles are called* tangent.

Proof. By Theorem 25, radiuses O_1A and O_2A are both perpendicular to m at A; since there can only be one perpendicular line to m at given point, it means that O_1, O_2, and A are on the same line, and that m is perpendicular to O_1O_2 at A.

Now, suppose that C_1, C_2 intersect at point $B \neq A$. Then by the previous theorem, $\overline{AB} \perp \overline{O_1O_2}$, therefore both $\overline{AB}$ and m are perpendicular to $\overline{O_1O_2}$ through A. We must therefore have that B is on m, but m is tangent to C_1 through A, thus has only one intersection with C_1, which is a contradiction. □

INSCRIBED ANGLE THEOREM

Consider a circle C with center O, and an angle formed by two rays from O. Then these two rays intersect the circle at points A, B, and the portion of the circle inside this angle is called the arc subtended by $\angle AOB$.

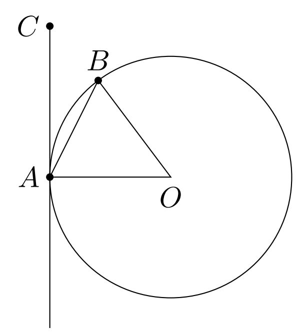


Theorem 29. *Let A, B, C be on a circle S with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in S.*

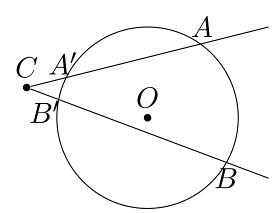
Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.
Case 1. Draw in segment $\overline{OC}$. Denote $m\angle A = x$, $m\angle B = y$. Since $\triangle AOC$ is isosceles, $m\angle AC) = x$; similarly $m\angle BCO = y$, so $m\angle ACB = x + y$, and $m\angle AOC = 180° - 2x$, $m\angle BOC = 180° - 2y$. Therefore, $m\angle AOC + m\angle BOC = 360° - 2(x + y)$. This implies $m\angle AOB = 2(x + y)$. □

Corollary. *If point C is on a circle with diameter AB, then $m\angle ACB = 90°$*

HOMEWORK



1. (Angle Theorems) Let's study Theorem 29 in a bit more detail!
   (a) Prove the converse of Theorem 29: namely, if a circle S is centered at O and A, B, are on S, and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on S. [Hint: let C' be the point where line AC intersects S. Show that then, $m\angle ACB = m\angle AC'B$, and show that this implies $C = C'$.]
   (b) Let A, B be on circle S centered at O and m the tangent to S at A, as shown on the right. Let C be on m such that C is on the same side of $\overleftrightarrow{OA}$ as B. Prove that $m\angle BAC = \frac{1}{2}m\angle BOA$. [Hint: extend $\overline{OA}$ to intersect S at point D so that $\overline{AD}$ is a diameter of S. What arc does $\angle DAB$ subtend?]
2. Here is a modification of Theorem 29.
   Consider a circle S and an angle whose vertex C is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\overgroup{AB}$ and $\overgroup{A'B'}$.
   Prove that in this case, $m\angle C = \frac{1}{2}(\overgroup{AB} - \overgroup{A'B'})$.
   [Hint: draw line AB' and find first the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]



3. Can you suggest and prove an analog of the previous problem, but when the point C is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vertical angles)?
4. Prove that a quadrilateral $ABCD$ can be inscribed in a circle if and only if sum of opposite angles is $180°$.