MATH 8
ASSIGNMENT 15: EUCLIDEAN GEOMETRY 4
JAN 25, 2026

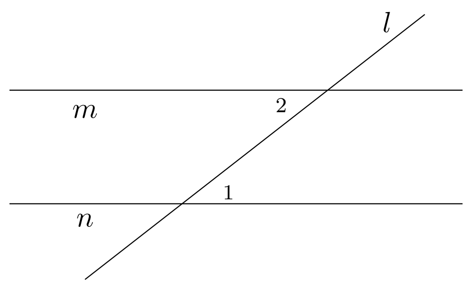
1. Axioms

Axiom 1. *For any two distinct points A, B, there is exactly one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.*

Axiom 2. *If distinct points A, B, C are on the same line, exactly one is* between the other two*; if point B is between A and C, then $AC = AB + BC$.*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*

Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure to the right (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle A'B'C'$ have two congruent sides and a congruent angle between these sides, then the triangles are congruent: if $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\angle ABC \cong \angle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a congruent side between these angles, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

2. Theorems summary

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be vertical angles. Then $m\angle 1 = m\angle 3$.*

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Theorem 8. *Sum of angles of a triangle is equal to* $180°$.

Theorem 9 (Base angles equal). *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

Theorem 10. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*

Theorem 11. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Corollary. *Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$. Then for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.*

Theorem 12. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

Theorem 13 (The triangle inequality). *In $\triangle ABC$, we have $AB + BC > AC$.*

Theorem 14. *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

Theorem 15. *Any quadrilateral $ABCD$ is a parallelogram if any one of the following conditions is true. In this case, all other conditions are also true.*

- *its opposite sides are equal ($AB = CD$ and $AD = BC$),* ***OR***
- *two opposite sides are equal and parallel ($AB = CD$ and $AB \parallel CD$),* ***OR***
- *its diagonals bisect each other ($AM = CM$ and $BM = DM$, where $AC \cap BD = M$),* ***OR***
- *its opposing angles are equal ($\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$).*

Theorem 16. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

Theorem 17. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

Theorem 18. *Let $ABCD$ be a trapezoid, with bases AD and BC, and let EF be the midline (i.e. E, F be midpoints of sides AB, CD respectively).*
Then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.

Theorem 19. *Let $ABCD$ be a rectangle. Then it is a parallelogram; in particular, the opposite sides are equal. Moreover, the diagonals are equal.*

3. Special points in a triangle

Recall the following definitions.

Definition.

- An *angle bisector* of angle $\angle BAC$ is the ray $\overrightarrow{AD}$ which is inside the angle and such that angles $\angle BAD$, $\angle CAD$ are congruent.
- A *perpendicular bisector* of segment AB is the line l which contains the midpoint M of AB and such that $l \perp AB$.
- An an *altitude* of a triangle $\triangle ABC$ is the perpendicular from a vertex to the line containing the oppoiste side, e.g. a segment AD such that $D \in \overleftrightarrow{BC}$, $AD \perp BC$. Note that the altitude may be outside the triangle.
- A *median* of a triangle $\triangle ABC$ is the segment connecting a vertex with the midpoint of the opposite side.

Note that while we defined the altitude to be the segment, commonly people would also say "altitude" meaning the the line containing that segment; same applies to medians in a triangle. Please always clarify which one you mean.

We had also proved in the homework the following results

Theorem 20. *Given two distinct points A, B, a point P is equidistant from them (i.e. $AP = BP$) if and only if P lies on the perpendicular bisector of AB.*

Theorem 21. *Given an angle $\angle BAC$ and point P inside this angle, P is equidistant from the two sides of the angle if and only if P lies on the angle bisector of $\angle BAC$.*

Using that, we can prove the following.

Theorem 22. *In a triangle $\triangle ABC$,*

1. *The three angle bisectors intersect at a single point inside the triangle; this point is called the* incenter *and is at equal distance from all three sides*
2. *The three perpendicular bisectors to the sides intersect at a single point; this point is called the* circumcenter *and is at equal distance from all three vertices.*
3. *The lines containing the three altitudes of the triangle intersect at a single point; this point is called the* orthocenter *of the triangle*
4. *The three medians intersect at a single point inside the triangle; this point is called the* centroid, *or center of mass, of the triangle. It divides each of the medians in the proportion* $2 : 1$.

Proof. The first three statements have been proved in the homeworks. We skip the proof of the lst one, as it is much easier done using the language of vectors whcih we will introduce later. ☐

CIRCLES

Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $OP = r$.

Given a circle with center O,

- A radius is a line segment from O to a point A on the circle
- A chord is a line segment between distinct points A, B on the circle
- A diameter is a chord that passes through O,

Recall that if O is equidistant from points A, B, then O must lie on the perpendicular bisector of AB (see Homework 13). We can restate this result as follows.

Theorem 23. *If AB is a chord of a circle, then the center O of this circle lies on the perpendicular bisector of AB.*

RELATIVE POSITIONS OF LINES AND CIRCLES

Theorem 24. *Let C be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular OP from O to l. Then:*

- *If $d > r$, then C and l do not intersect.*
- *If $d = r$, then C intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is* tangent *to λ at P.*
- *If $d < r$, then C intersects l at two distinct points.*

Proof. First two parts easily follow from corollary to Theorem 11: slant line is longer than the perpendicular.

For the last part, it is easy to show that C can not intersect l at more than 2 points (see homework problem 1). Proving that it does intersect l at two points is quite hard; in fact, it can not be deduced from the axioms we have – we need some axioms basically saying that there are no "holes" in the real line. We omit that. □

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

Theorem 25. *Let C be a circle with center O, and let l be a line through a point A on C. Then l is tangent to C if and only if $l \perp \overleftrightarrow{OA}$*

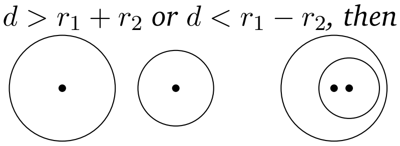
Proof. By definition, if l is the tangent line to C, then it has only one common point with C, and this point is the base of the perpendicular from O to l; thus, OA is the perpendicular to l.

Conversely, if $OA \perp l$, it means that the distance from l to O is equal to the radius (both are given by OA), so l is tangent to C. □

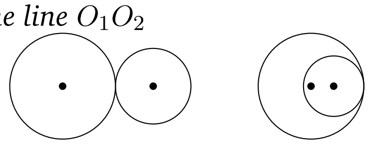
Similar results hold for relative position of a pair of circles. We will only give part of the statement.

Theorem 26. *Let C_1, C_2 be two circles, with centers O_1, O_2 and radiuses r_1, r_2 respectively; assume that $r_1 \geq r_2$. Let $d = O_1O_2$ be the distance between the centers of the two circles.*

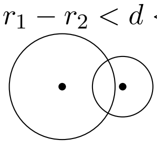
- *If $d > r_1 + r_2$ or $d < r_1 - r_2$, then these two circles do not intersect.*



- *If $d = r_1 + r_2$ or $d = r_1 - r_2$ then these two circles have a unique common point, which lies on the line O_1O_2*



- *If $r_1 - r_2 < d < r_1 + r_2$, then the two circles intersect at exactly two points.*



We skip the proof.

Definition. Two circles are called tangent if they intersect at exactly one point.

HOMEWORK

1. Let A, B, C be three distinct points which are not on the same line. Show that then, there is a unique circle that contains these points; the center of this circle must be the intersection point of the three perpendicular bisectors to the sides of $\triangle ABC$. This circle is called the circumscribed circle of $\triangle ABC$
2. Let A, B, C be three distinct points which are on the same line. Show that then, there is no circle that contains these points. [Hint: if such a circle existed, its center must lie on perpendicular bisector fo AB and also on the perpendicular bisector of BC.]
3. Show that if a circle is tangent to both sides of the angle $\angle ABC$, then the center of that circle must lie on the angle bisector. [Hint: this center is equidistant from the two sides of the circle.] Show that conversely, given a point O on the angle bisector, there exists a circle with center at this point which is tangent to both sides of the angle.

4. Use the previous problem to show that for any triangle, there is a unique circle that is tangent to all three sides. This circle is called the inscribed circle of $\triangle ABC$

5. Let two circles C_1, C_2 be both tangent to a line l at the same point $A \in l$. Show that then these two circles are tangent to each other, i.e. have exactly one common point A.