MATH 8
ASSIGNMENT 11: EUCLIDEAN GEOMETRY 1: AXIOMS.
DEC 7, 2025

Euclidean geometry is a way describe geometric properties of various figures in the plane *exactly*. Figures are understood as sets of **points**; we will use capital letters for points and write $P \in m$ for *"point P lies in figure m"*, or *"figure m contains point P"*. A **point** is the simplest geometric object and is so basic that it can not be explained in terms of even simpler notions; everyone is believed to know what a "point" is. In addition, there are some other basic notions (**lines**, **distances**, **angle measures**) that are not be defined, and understood as elementary. Instead, we can state some basic properties of these objects; these basic properties are usually called **POSTULATES** or **AXIOMS** of Euclidean geometry. Using these basic properties, we can then apply rules of logic to deduce other properties (lemmas, theorems). **All results in Euclidean geometry should be proven by deducing them from the axioms**; justifications *"it is obvious"*, *"it is well-known"*, or *"it is clear from the picture"* are not acceptable! In addition to logical rules, we will also use all the usual properties of real numbers, equations, inequalities, and alike.

Note that we will make diagrams helping us understand relations between points, lines, angles, etc. Diagrams are *never perfect* (and do not have to be) because we use them only as aid to our thought process. Unlike diagrams, statements made in geometry are *exact*, and thus require more care to demonstrate than just a picture. For example, it is impossible to draw two precisely parallel lines; we can only "pretend" that their finite segments represent them in a picture, but we have to use axioms (and theorems already proven) to demonstrate additional properties. On its own, a picture is useful *only as illustration* to help understand our argument.

For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's *Elements*, dated about 300 BC, and used as the standard textbook for the next 2000 years. Nowadays it is available online at `http://math.clarku.edu/~djoyce/java/elements/toc.html`

1. Basic objects

These basic objects and notions are the basis of all our later constructions: we will define and discuss all other objects in terms of the basic ones. We do not need (and cannot give) definitions for these basic objects, and consider them self-evident, as well as some of their properties.

- Points (denoted by upper-case letters: A, B, $\ldots$) can be said to have zero "size" in any respect;
- Lines (denoted by lower-case letters: $l, m, \ldots$): infinite in both directions and split the plane into "half-planes";
- Distances: for any two points A, B, there is a non-negative number AB, called the distance between A, B. The distance is zero *if and only if* points exactly coincide.

Note that one can measure distances with a ruler and angles with a protractor, but only as precisely (or imprecisely) as the tool allows. In geometry, however, these measures are considered **exact numbers**.

We will also frequently use words *"between"* when describing relative position of points on a line (as in: A is between B and C) and *"inside"* (as in: point C is inside angle $\angle AOB$). We do not give a full list of axioms for these notions; it is possible, but rather boring.

Having these basic notions, we can now define more objects. Namely, we can give definitions of

- interval, or line segment (notation: $\overline{AB}$): set of all points on line $\overleftrightarrow{AB}$ which are between A and B, together with points A and B themselves. The segment length (denoted as AB or $|AB|$) is the distance between A and B.

- ray, or half-line (notation: $\overrightarrow{AB}$): set of all points on the line $\overleftrightarrow{AB}$ which are on the same side of A as B (Note that we have not defined the concept *"on the same side"* but will be using it in the future).
- angle (notation: $\angle AOD$): figure consisting of two rays ($\overrightarrow{OA}$ and $\overrightarrow{OD}$) with a common vertex (O). For any angle $\angle ABC$, there is a non-negative real number $m\angle ABC$, called the measure of this angle.
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself (it is a rather convenient convention, which will make our lives easier – the intuition here is that parallel lines have the same "direction"). Also, the angle between two parallel rays is zero.

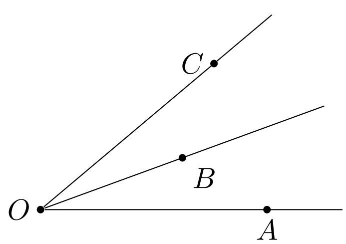
2. First axioms

After we introduced some objects, including undefined ones, we need to have statements (*axioms*) that describe their properties. Of course, the lack of definition for undefined objects makes such properties impossible to prove. The goal here is to state the *minimal number* of such properties that we take for granted, just enough to be able to prove or derive harder and more complicated statements. Here are the first few axioms:

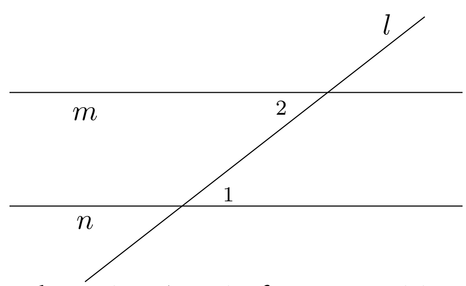
Axiom 1. *For any two distinct points A, B, there is exactly one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.*

Axiom 2. *If distinct points A, B, C are on the same line, exactly one is* between the other two*; if point B is between A and C, then $AC = AB + BC$.*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

3. First theorems

Now we can proceed with proving some results based on the axioms above.

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Proof. Proof by contradiction: Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more than one point) must be false. □

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Proof. Here we have to prove two things: the existence of a parallel line through the given point not on the given line, and its uniqueness. Below we provide a sketch of the proof – please fill in the details and draw a diagram at home!

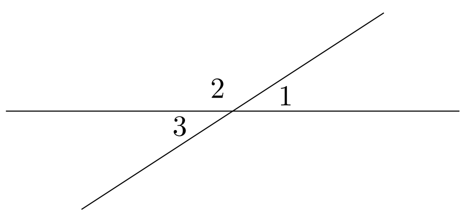
Existence: Let m be any line that goes through P and intersect l at point O. Let A be a point on the line l. Then we can measure the angle $\angle POA$. Now, let PB be such that $m\angle BPO = m\angle POA$ and B is on the other side of m than A. In this case, by Axiom 4, $\overleftrightarrow{AB} \| \, l$.

Uniqueness: Imagine that there are two lines m, n that are parallel to l and go through P. Take a line k that goes through P and intersects l in point O. Let A be a point on line l distinct from O, and B, C — points on lines m and n respectively on the other side of line k than A. Since both m, n are parallel to l, we can see that $m\angle AOP = m\angle BPO = m\angle CPO$ – but that would mean that lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{CP}$ are the same — contradiction to our assumption that there are two such lines. □

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Proof. Assume that l and n are not parallel and intersect at point P. But then it appears that there are two lines that are parallel to m are go through point P — contradiction with Theorem 2. □

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Proof. Left as a homework exercise. □

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*

Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Proof. Left as a homework exercise. □

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

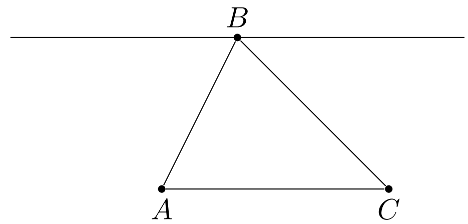
Proof. Left as a homework exercise. □

4. Triangles

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a triangle, written $\triangle ABC$. The three respective angles are called the triangle's interior angles.)*

Proof.

The proof is based on the figure below and use of Alternate Interior Angles axiom. Details are left to you as a homework.



□

HOMEWORK

1. Consider the following "geometry":
- Points: points on the unit sphere in 3 dimensional space
- Lines: "big circles" on the sphere (i.e. circles that are obtained as intersection of the sphere with a plane through the center of the sphere)
- Distances: measured along the arcs of big circles

Which axioms of Euclidean geometry hold in this new "geometry"? which fail?

2. The following logic and geometric statements come in equivalent pairs. Each logic statement has exactly one geometric statement that is equivalent to it. Match these statements into their equivalent pairs, with an explanation of why the pairs you chose are equivalent. [**Note**: the quantifier $\exists!$ stands for "there exists a unique...", and $\emptyset$ is an empty set.]

Geometric statements:

(a) For any two distinct points there is a unique line containing these points.

(b) Given a line and a point not on the line there exists a unique line though the given point that is parallel to the given line.

(c) If two lines are parallel and another line intersects one of them, then it intersects the other one as well.

(d) If two lines are parallel to the same line, then they are parallel to each other

Logic statements:

(a) $\forall l\ \forall m : l \parallel m \implies [\forall n\ (n \cap l \neq \emptyset \rightarrow n \cap m \neq \emptyset)]$

(b) $\forall A\ \forall B : A \neq B \implies [\exists!\ l\ (A \in l \wedge B \in l)]$

(c) $\forall l\ \forall m\ [(\exists n : n \parallel l \wedge n \parallel m) \rightarrow (l \parallel m)]$

(d) $\forall l\ \forall A : A \notin l \implies [\exists!\ m\ (A \in m \wedge m \parallel l)]$

3. (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.

(a) Study the proof of Theorem 2 and draw a diagram that illustrates it.

(b) Study the proof of Theorem 3.

(c) Prove Theorem 5.

(d) Prove Theorem 6.

(e) Prove Theorem 7.

4. Complete the proof of Theorem 8, about sum of angles of a triangle.

5. What is the sum of angles of a quadrilateral? of a pentagon?

6. Let D be a point on continuation of side AC of triangle $\triangle ABC$ as shown to the right. Show that then $m\angle BCD = m\angle A + m\angle B$. (Angle $\angle BCD$ is called an exterior angle of $\triangle ABC$).