Homework 9: Pascal's triangle, applications

HW9 is Due Nov 23.

1. **Pascal's triangle and the binomial coefficients.**

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}
$$

Every entry in it is obtained as the sum of two entries above it. The k-th entry in the n-th line is denoted by $\binom{n}{k}$, or by nC_k . Note that both n and k are counted from 0, not from 1:
for example, $\binom{6}{2} = 15$, $\binom{3}{0} = 1$. Now let us think about other applications of these numbers.

2. **Example applications of the binomial coefficients**

a) Chessboard paths

In the previous handout we saw that these numbers appear in a problem about counting paths from the lower left corner of the board to the upper right corner. We observed the following:

$\binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k units to the right.

For example, the number of paths that go to the upper right corner of a 6 × 6 board is equal to $\binom{10}{5}$, as *each* such path must have 5 steps to the right and 5 steps up, taken in any order. This means that we have total of n = 10 steps made of k = 5 steps up and the rest, 10 - 5 = 5 steps to the right. Or, from 10 possible moves we pick 5 to be up.

b) Words with Rs and Us or 1s and 0s

Each of these paths on the board going up and to the right can be written as a sequence of steps; let R be a step to the right, and U a step up. For example, a possible path is: RRRRRUUUUU will go five steps to the right and five steps up, eventually ending in the upper right corner of a 6 × 6 board. Another possible sequence is RRUUURRRUU. There is a correspondence between paths of length **n** and strings of length **n** consisting of ***Rs*** and ***Us*** only. Now let us switch Rs to 0s and Us to 1s. We already know that $\binom{n}{k}$ = is the number of paths going k units up and n − k units to the right. This corresponds to words of length n with k Us and n − k Rs, which is the same as the number of strings of length n with k 1s and n − k Us. We have the following result:

$\binom{n}{k}$ = the number of words with length n that can be written using k ones and n − k zeroes or k Us and n-k Rs.

c) Combinations: *the choice of **k** things from a set of **n** things without repetition ("replacement") and where the order does not matter.*

Consider now a set on n elements; let's number them from 1 to n. Then, for each string of length n with 0s and 1s, we can select those elements that correspond to 1s and omit those elements that correspond to 0s. In this way, we will get <u>a subset of size k</u>. This is another property of the binomial coefficients:

$\binom{n}{k}$ = the number of ways to choose k items out of n (order doesn't matter)

3. Summary: binomial coefficients represent

$nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right
= the number of words that can be written using k ones and n − k zeroes
= the number of ways to choose k items out of n (order doesn't matter)

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

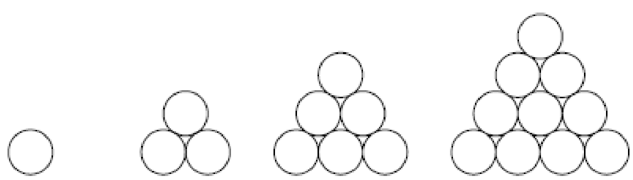
Note: In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use the Pascal's triangle to find the value of $\binom{n}{k}$, where k is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. Which of the numbers in Pascal triangle are even? Can you *guess the pattern*, and then carefully explain why it works? (no formulas)
2. What is the sum of all entries in the n-th row of Pascal triangle? Try computing the first several answers and then guess the general formula.
3. What is the alternating sum of all the numbers in n-th row of Pascal triangle, i.e.

$$1 + \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \ \ldots$$

   Try computing the first several sums and then guess the general formula.

4. Let us draw a figure consisting of n rows of circles as shown in the figure below (for n = 1, 2, 3, 4):



   Let T_n be the number of circles in n-th figure (for example, $T_1 = 1, T_2 = 3, T_3 = 6 \ldots$). These numbers are sometimes called the triangular numbers.

   (a) What is the difference $T_{n+1} - T_n$? Try for a few n as 1, 2, 3, 4, 5, …

   (b) Show that the numbers T_n appear in the Pascal triangle as shown below, that is $T_n = \binom{n+1}{2}$. Again, try n = 1, 2, 3, 4 …

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & \boxed{1} & & \\
 & 1 & & 3 & & \boxed{3} & & 1 & \\
1 & & 4 & & \boxed{6} & & 4 & & 1 \\
 & & & & \dots & & & &
\end{array}
$$

5. How many “words” of length 5 one can write using only letters U and R, namely 3 U’s and 2 R’s? What if you have 5 U’s and 3 R’s? [Hint: each such “word” can describe a path on the chessboard, U for up and R for right. . .]
6. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?
7. If we toss a coin 10 times, what is the probability that all will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?
8. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub — and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will move:
   a) 10 steps north
   b) 10 steps south
   c) 4 steps north
   d) will come back to the starting position
9. If you have a group of 4 people, and you need to choose one to go to a competition, how many ways are there to do it? if you need to choose 2? if you need to choose 3?
10. In a meeting of 25 people, each much shake hands with each other. How many handshakes are there altogether?