Homework 6: Arithmetic and Geometric sequences

HW6 is Due Nov 2nd.

1. Arithmetic sequence (progression)

A sequence of numbers is an arithmetic sequence if the difference between consecutive elements is the same number. This number is called a common difference, d.

For example: 1, 5, 9, 13, 17, ... The difference here is d = 4.

Sequence elements (terms) are labeled according to their position in the sequence using a counter **n** as a subscript. The value of the n-th element in a sequence is labeled as $\mathbf{a_n}$. Then, the first term in the sequence has n = 1 and a value of $a_1 = 1$, the second element is $a_2 = 5$, and so on.

We could find any element of a sequence knowing the first element a_1 and the difference d.
For example, what is a_{100}?

$a_1 = 1$
$a_2 = a_1 + d = 1 + 4 = 5$
$a_3 = a_2 + d = a_1 + 2d = 1 + 2 \times 4 = 9$
$a_4 = a_3 + d = a_1 + 3d = 1 + 3 \times 4 = 13$
…
$\boldsymbol{a_n = a_1 + (n-1)d}$

So $a_{100} = a_1 + 99d = 1 + 99 \times 4 = 397$

2.Property of an arithmetic sequence

A property of an arithmetic sequence is that any term is the arithmetic mean of its neighbors.

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Proof:

$$a_n = a_{n-1} + d$$
$$a_n = a_{n+1} - d$$

Add the left and the right sides:

$$2a_n = (a_{n-1} + d) + (a_{n-1} - d)$$
$$2a_n = a_{n-1} + a_{n-1}$$

Dividing by 2:

$$a_n = \frac{a_{n-1} + a_{n-1}}{2}$$

Another property of arithmetic sequences is that we can find the common difference d if we know any 2 terms a_s and a_t

$$d = \frac{a_s - a_t}{s - t}$$

3. Sum of an arithmetic sequence

$$S = a_1 + a_2 + a_3 + \cdots + a_n = n \times \frac{a_1 + a_n}{2}$$

Proof: we write the sum in 2 ways, in increasing order and in decreasing order:

$S = a_1 + a_2 + a_3 + \cdots + a_n$
$S = a_n + a_{n-1} + a_{n-2} + \cdots + a_1$

Adding up left and right sides:

$$2S = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots$$

We notice that:

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots$$

$$2S = (a_1 + a_n) \times n$$

$$S = \frac{(a_1 + a_n) \times n}{2}$$

4. Arithmetic sequences -summary

$$a_n = a_1 + (n-1)d$$
$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$
$$d = \frac{a_s - a_t}{s - t}$$
$$S = \frac{(a_1 + a_n) \times n}{2}$$

Geometric Sequence:

1. Geometric sequence (progression)
A sequence of numbers is a geometric progression if the next number in the sequence is the current number times a constant called the **common ratio**, let's call it **q**.

For example: 6, 12, 24, 48, ... The common ratio here is q = 2.

Sequence elements (terms) are labeled according to their position in the sequence using a counter **n** as a subscript. The value of the n-th element in a sequence is labeled as $\mathbf{a_n}$**.** Then, the first term in the sequence has n = 1 and a value of $a_1 = 6$, the second element is $a_2 = 15$, and so on.

We could find any element of a sequence knowing the first element a_1 and the ration q.
Example: *What is b_{10}? What is the n^{th} term?*

$$a_1 = 6$$

$$a_2 = a_1 \times q = 6 \times 2 = 12$$

$$a_3 = a_2 \times q = (a_1 \times q) \times q = a_1 \times q^2 = 6 \times 2^2 = 24$$

$$a_4 = a_3 \times q = (a_1 \times q^2) \times q^2 = a_1 \times q^3 = 6 \times 2^3 = 48$$

....

$$\mathbf{a_n = a_1 \times q^{n-1}}$$

So $a_{10} = a_1 \times q^9 = 6 \times 2^9 = 6 \times 512 = 3072$

2.Property of a geometric sequence
A property of a geometric sequence is that any term is geometric mean of its neighbors or any two equally distanced neighbors.

$$a_n = \sqrt{a_{n-1}.a_{n+1}} = \sqrt{a_{n-\mathrm{k}}.a_{n+\mathrm{k}}}$$

Proof:

$$a_n = a_{n-1} \times q$$

$$a_n = a_{n+1} \div q$$

Multiplying these two equalities gives us:

$$a_n^2 = a_{n-1} \cdot a_{n+1}$$

from where we can get what we need.

3. **Sum of a geometric sequence [We will discuss this next week],**

a) Sum of the first n-terms:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = a_1 \times \frac{(1 - q^n)}{1 - q}$$

Proof: To prove this, we write the sum and we multiply it by q:

$$S = a_1 + a_2 + a_3 + \cdots \mathrm{a}_{n-1} + a_n$$
$$qS = qa_1 + \mathrm{q}a_2 + \mathrm{q}a_3 + \cdots + qa_{n-1} + qa_n$$

Remember that $qa_{n-1} = a_n$, so that the last term is $qa_n = q \times (a_1 \times q^{n-1}) = a_1 \times q^n$:

$$S = a_1 + a_2 + a_3 + \cdots \mathrm{a}_{n-1} + a_n$$
$$qS = a_2 + a_3 + a_4 + \cdots + a_n + a_{n+1}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$S_n - qS_n = (a_1 - a_{n+1}), \quad \text{or}$$

$$S_n(1 - q) = (a_1 - a_1 q^n)$$

$$S_n(1 - q) = a_1(1 - q^n)$$

from which we get the formula above.

b) Sum of **Infinite Sum**

If $0 < q < 1$, then the sum of the geometric progression is approaching some numbers, which we can call a **sum of an infinite geometric progression**, or just an **infinite sum**.

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ \ldots \ = \ 2$$

The formula for the infinite sum is the following: $\mathrm{S} = \frac{a_1}{1-q}$

4. Geometric sequences -summary

$$a_n = a_1 \times q^{n-1}$$

$$a_n = \sqrt{a_{n-1} . a_{n+1}}$$

$$S_n = a_1 \times \frac{(1 - q^n)}{1 - q}$$

$$\mathrm{S} = \frac{a_1}{1 - q}$$

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Find the sum of the first 10 terms for the series: 4, 7, 10, 13, . . .
2. Find the sum of the first 1000 odd numbers.
3. Find the sum 2 + 4 + · · · + 2018.
4. There are 25 trees at equal distances of 5 meters in a line with a well, the distance of the well from the nearest tree being 10 meters. A gardener waters all trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.
5. * An arithmetic progression has first term $a_1 = a$ and common difference $d = -1$. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n.
6. * The sum of the first 20 terms of an arithmetic progression is 200, and the sum of the next 20 terms is -200. Find the sum of the first hundred terms of the progression.
7. Write the first 5 terms of a geometric progression if $a_1 = -20$ and $q = \frac{1}{2}$
8. What are the first 2 terms of the geometric progression: a_1, a_2, 24, 36, 54, ...?
9. What is the common ratio of the geometric progression: $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$? What is a_{10}? What is a_{100}?
10. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50-th term?