Homework 16: Coordinate geometry II
HW Due on March 8th 2026

1. Coordinate geometry: Introduction

In this section of the course, we are going to study coordinate geometry. The basic notion is the **coordinate plane**– a plane with a given fixed point, called the **origin**, as well as two perpendicular lines – **axes**, called the x-**axis** and the y-**axis**. x-axis is usually drawn horizontally, and y-axis — vertically. These two axes have a **scale** – "distance" from the origin.

The scales on the axes allow us to describe any point on the plane by its **coordinates**. To find coordinates of a point P, draw lines through P perpendicular to the x- and y-axes. These lines intersect the axes in points with coordinates x_0 and y_0. Then the point P has x-coordinate x_0, and y-coordinate y_0, and the notation for that is: $P(x_0, y_0)$.

The **midpoint** M of a segment AB with endpoints A(x_1, y_1) and B(x_2, y_2) has coordinates: $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

2. Lines

Given some relation which involves variables x, y (such as $x + 2y = 0$ or $y = x^2 + 1$), we can plot on the coordinate plane all points $M(x, y)$ whose coordinates satisfy this equation. Of course, there will be infinitely many such points; however, they usually fill some smooth line or curve. This curve is called the **graph** of the given relation.

Every relation (**equation**) of the form: $y = \boldsymbol{m}x + \boldsymbol{b}$

where $\boldsymbol{m}, \boldsymbol{b}$ are some numbers, defines <u>a straight line</u>. The slope of this line is determined by m: as you move along the line, y changes m times as fast as x, so if you increase x by 1, then y will increase by m. In other words, given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ **slope** can be computed by dividing change of y: $\Delta y = y_2 - y_1$ by the change of x: $\Delta x = x_2 - x_1$:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

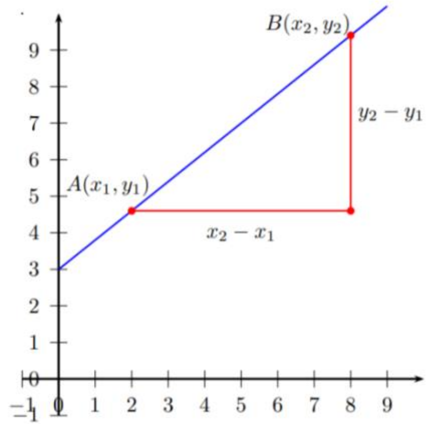


Fig. 1

Two non-vertical lines are **parallel** if and only if they have the **same slope**.
In the equation $y = mx + b$, b is the y-**intercept**, and determines where the line intersects the vertical axis (y-axis). The equation of the **vertical** line is $x = k$, and the equation of the **horizontal** line is $y = k$. Notice that in case of the vertical line, the slope is undefined.

3. Distance between points. Circles.

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the following formula:

$$\boldsymbol{d} = \sqrt{(\boldsymbol{x_2} - \boldsymbol{x_2})^2 + (\boldsymbol{y_2} - \boldsymbol{y_1})^2}$$

This formula is a straightforward consequence of Pythagoras' Theorem (Fig. 1).

The equation of the circle with the center $M(x_0, y_0)$ and radius $\boldsymbol{r}$ is:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

This equation means, that points (x, y) should be at distance $\boldsymbol{r}$ from the given point $M(x_0, y_0)$.

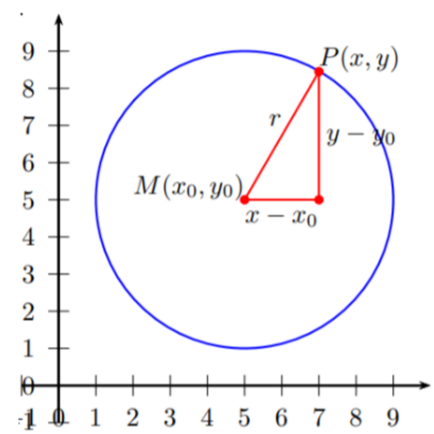


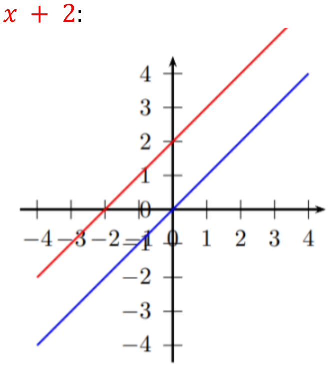
Fig.2

4. Graphs of functions

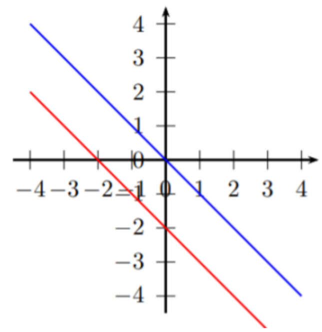
In general, the relation between x and y could be more complicated and could be given by some formula of the form $y = f(x)$, where f is some function of x (i.e., some formula which contains x). Then the set of all points whose coordinates satisfy this relation is called the **graph** of f.

Line. The graph of the function $y = mx + b$ is a straight line. The coefficient $\boldsymbol{m}$ is called the *slope*.

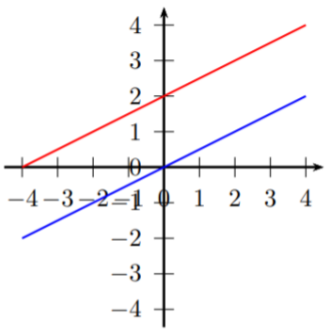
$y = x$; $y = x + 2$:



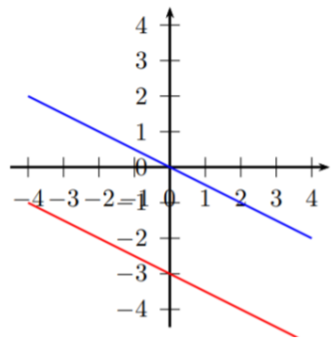
$y = -x$; $y = -x - 2$:



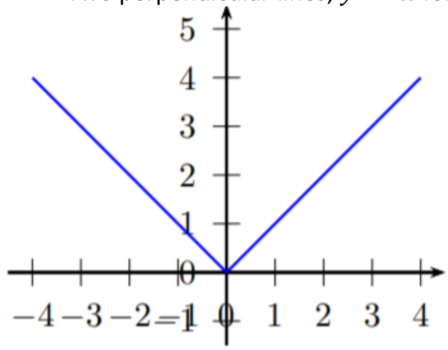
$y = \frac{1}{2}x$; $y = \frac{1}{2}x + 2$:



$y = -\frac{1}{2}x$; $y = -\frac{1}{2}x - 3$:



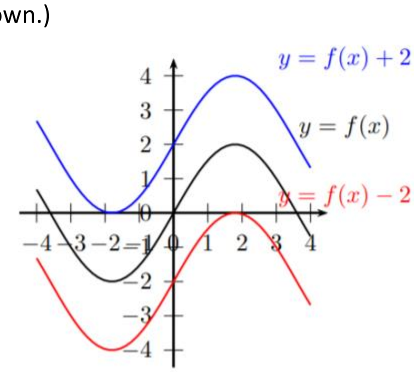
Absolute value of a line. $\boldsymbol{y = |x|}$ Two perpendicular lines, $y = x$ for $x > 0$ and $y = -x$ for $x < 0$.

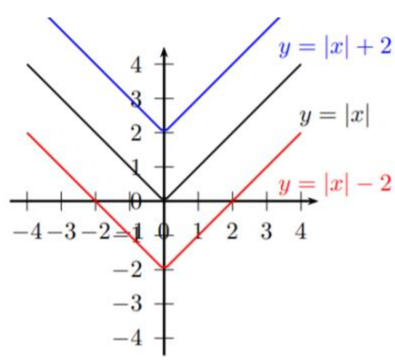


5. Function transformations

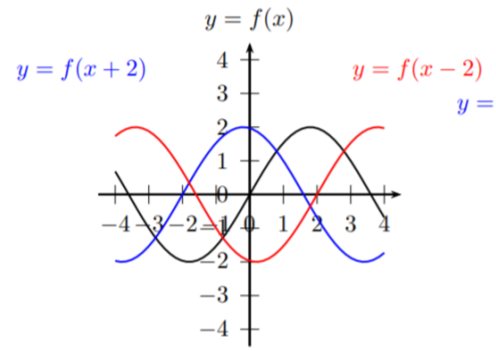
Having learned a number of basic graphs, we can produce new graphs, by doing certain transformations of the equations. Here are two of them.

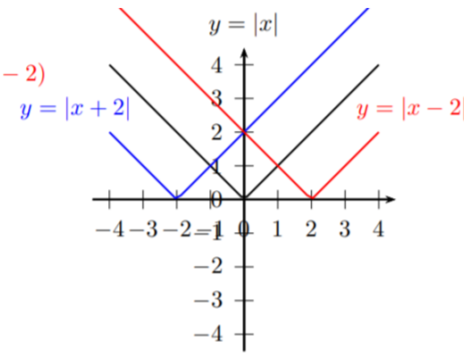
Vertical translations: Adding constant $\boldsymbol{c}$ to the right-hand side of equation shifts the graph by $\boldsymbol{c}$ units up (if c is positive; if c is negative, it shifts by |c| down.)





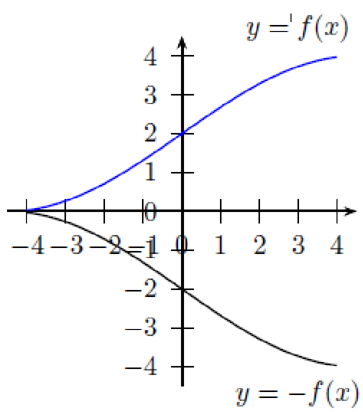
Horizontal translations: Adding constant ***c*** to x shifts the graph by ***c*** units left if ***c*** is positive; if ***c*** is negative, it shifts by ***c*** right.



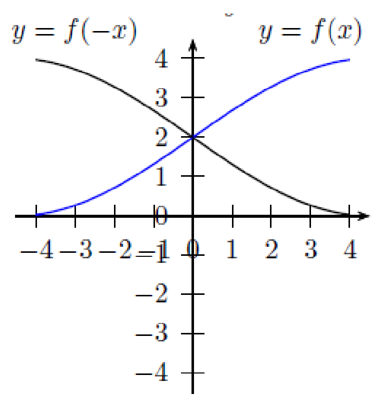


Reflection

Multiplying the function by −1 reflects the graph around the x-axis:



Replacing in the equation x by −x reflects the graph around the y-axis:



Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

ALL GRAPHS/POINTS/FIGURES SHOULD BE DRAWN BY YOU - NOT PRINTED!   USE QUADRILE PAPER!

1. Let ***C*** be the circle with center at (0, 1) and radius 2, and ℓ- the line with slope 1 going through the origin. Find the intersection points of the circle ***C*** and line ℓ, and compute the distance between them.

2. There are two lines defined in a plane.
   a. What are the possible orientations of those two lines and how many crossing points could they have? Please explain.

b. Sketch the graphs of the functions $y = |x + 1|$ and $y = -x + 0.25$. How many crossing points do they have?
c. How many solutions do you think this equation has?

$$|x + 1| = -x + 0.25$$

Note: just answer how many are there. You are solving graphically – look at your graph. Justify your answer about the number of solutions.

d. (*) Solve the above equation algebraically. What is the meaning of the solution (x, y)?

3. The distance from a point P = (u, v) to a line expressed in a standard form as $ax + by + c = 0$ is given by the formula:

$$\text{(Eq.1)} \qquad d = \frac{au+bv+c}{\sqrt{a^2+b^2}}$$

(You could try to prove this formula for the case when intercept $c = 0$ if you wish to challenge yourself.) Here, we will just use the formula to calculate the distance between a line and a point and then check the value by graphing the line and the point and calculating the distance from the graph.

a. If the line ℓ_1 is defined by $3x + 6y = 0$ and the point P is at $(5,4)$. Using the formula in Eq.1, calculate the distance between P and the line.
b. Draw a line ℓ_1 defined by $3x + 6y = 0$ (express the equation in slope – intercept form) and the point P = (5,4). Find the equation of the line ℓ_2 which passes through P and is perpendicular to the line ℓ_1. Find the point M where the two lines cross (graphically or algebraically). Calculate the distance PM using the coordinates of the two points. Is this value the same as the one you found in (a)?

4. Sketch the graphs of the following functions. Then, shift each function 2 units up, 2 units right, and reflect with respect to the x-axis. Sketch the result and write the equation.
   a. $y = x$
   b. $y = x^2$
   c. $y = 1/x$
   d. $y = \sqrt{x}$

Example: $a)\ y = x + 2,\ \ y = (x - 2) + 2,\ \ y = -[(x - 2) + 2] = -(x - 2) - 2$

5. In **desmos**, draw each graph. Is this what you expected to see? Do not attach, just think why you got that result. How is c) connected to
   a. Draw the graph of the equation: $x^2 + y^2 - 1 = 0$.
   b. Draw the graph of the equation: $xy = 0$. *What does this do? It defines something!*
   c. Draw the graph of the equation: $x^2 + y^2 = 0$. *What does this do? It defines something!*
   d. Draw the graph of the equation: $(x^2 + y^2 - 1)(x^2 + (y - 1)^2 - 1) = 0$. *How is this connected to b)?*
   e. Draw the graph of the equation: $(x^2 + y^2 - 1)^2 + (x^2 + (y - 1)^2 - 1)^2 = 0$. *How is this connected to d) ?*