## Homework 12: Vieta's formulas

### 1. Solving the complete quadratic equation

- By completing the square

"Completing the square" works by using the formulas for fast multiplication $(a \pm b)^2 = a^2 \pm 2ab + b^2$ (*)
Here is an example how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x+3)^2 - 7 = (x+3)^2 - \left(\sqrt{7}\right)^2 = \left(x+3+\sqrt{7}\right)\left(x+3-\sqrt{7}\right)$$

Thus, $x^2 + 6x + 2 = 0$ if and only if $(x + 3 + \sqrt{7}) = 0$, which gives $x = -3 - \sqrt{7}$, or if $(x + 3 - \sqrt{7}) = 0$, which gives $x = -3 + \sqrt{7}$.

- By using the quadratic formula

**Steps:** for the equation in the standard form $ax^2 + bx + c = 0$

List coefficients: $a = , b = , c =$
Find the determinant D: $\boldsymbol{D = b^2 - 4ac}$
Check the number of roots (solutions): The determinant, D, determines the number of solutions.
If D < 0, there are no real solutions; if D = 0, there is one solution, if D > 0, there are two solutions.
Find the solutions: $\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}}$

### 2. Vieta's formulas

Of course, with the quadratic formula we can always solve any quadratic equation, and then do the operations with its roots. However, sometimes it is unnecessary. The formulas below are called Vieta's Formulas and allow us to find the sum and the product of roots of a quadratic equation without explicitly calculating them. If you would like to use the formulas to find (guess) the roots, this will be easier if the coefficients $a, b, c$ are whole numbers.

**for equation in standard form:** $ax^2 + bx + c = 0$

The roots of the quadratic equation are related to the coefficients: $\boldsymbol{x_1 x_2 = \frac{c}{a}}$ and $\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$
In the special case when $\boldsymbol{a = 1}$, $\boldsymbol{x_1 x_2 = c}$ and $\boldsymbol{x_1 + x_2 = -b}$

How did we get to the Vieta's formulas? if $x_1$ and $x_2$ are the roots, then the quadratic equation can be written from standard into factored form:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

For the special case when a = 1, we set a = 1 and then multiplying the expressions in the right hand side, we get: $x^2 + bx + c = (x - x_1)(x - x_2) = x^2 - x_1 - x_2 x + x_1 x_2 = x^2 - (x_1 + x_2)x + x_1 x_2$

From comparing the numbers in front of the same powers of $x$ the coefficients we can get the following:

$$x_1 + x_2 = -b$$
$$x_1 x_2 = c$$

### 3. Generalized Vieta's formulas

In the previous section, we looked at Vieta's formulas for quadratic equations. That is, if $x_1, x_2$ are roots of quadratic polynomial (the quadratic equation written in a standard form) $ax^2 + bx + c = 0$

$$\boldsymbol{x_1 x_2 = \frac{c}{a}} \quad \text{and} \quad \boldsymbol{x_1 + x_2 = -\frac{b}{a}}$$

In addition to quadratic equations, we can also look at other types of equations:
Cubic equations: These are the equations with the 3rd power terms ($x^3$), generally written as

$$ax^3 + bx^2 + cx + d = 0$$

- 4-th power equations: These are the equations with the 4th power terms ($x^4$), generally written as

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

- other equation of higher power of x . . .

We are not going to study cubic and other equations of higher power now. It is sufficient to say that there are formulas for solving cubic equation (Cardanos Formula), and even formulas for solving equations of the 4th power - but are rarely used, and are pretty large. It can also be proven that equations of 5th power of higher do not have a formula, and it is impossible to find a formula for them.

Interestingly, Vieta formulas can be generalized for an equation of any higher power and they work for real and complex solutions. Similar to what we did for quadratic equations, if the equation of degree $n$

$$p(x) = ax^n + bx^{(n-1)} + cx^{(n-2)} + dx^{(n-3)} + \cdots + w = 0$$

has $n$ roots $x_1, x_2, \ldots, x_n$, then one can write it as: $p(x) = a(x - x_1) \ldots (x - x_n) = 0$
Expanding the right-hand side, we obtain Vieta formulas:

$$x_1 + x_2 + \cdots + x_n = -\frac{b}{a}$$

$$x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \cdots = \frac{c}{a}$$

$$x_1x_2x_3 + x_1x_2x_4 + \cdots + x_2x_3x_4 + \cdots = -\frac{d}{a}$$

$$\cdots$$

$$x_1x_2 \ldots x_n = (-)\frac{w}{a}$$

That is, in the generalized Vieta's formulas the sum of all roots is $-\frac{b}{a}$, the sum of all possible pairwise products of roots is $\frac{c}{a}$, etc., until we get to the product of all roots being equal to $\frac{w}{a}$ with an appropriate sign. Notice, the signs alternate.

4. **formulas for fast multiplication** $a^2 - b^2 = (a - b)(a + b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

## *Homework problems*

***Instructions:*** Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Let $x$ and $y$ be some numbers. Use the formulas for fast multiplication to rewrite the following expressions using only $(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{B}$ and $\boldsymbol{xy} = \boldsymbol{C}$, where B and C are just number. To do that, present the expressions as sums and/or products of $x$ and $y$, then substitute the sums/products with B and C. No $x$ and $y$ are allowed in the answers!

Example: $x^2 + y^2 = x^2 + y^2 + 2xy - 2xy = (x + y)^2 - 2xy = B^2 - 2C$
We completed the square using the formula: $a^2 \pm 2ab + b^2 = (a \pm b)^2$

a. $(x - y)^2 =$
b. $\frac{1}{x} + \frac{1}{y} =$
c. $\frac{1}{x-1} + \frac{1}{y-1} =$
d. $x^2 - y^2 =$
e. $x - y =$
Hint: use $x^2 - y^2 = (x + y)(\boldsymbol{x} - \boldsymbol{y})$
f. (*) $x^3 + y^3$
Hint: first compute $(x + y)(x^2 + y^2)$

2. Let $x_1$ , $x_2$ be the roots of the equation $x^2 + 5x - 7 = 0$. Using the Vieta's formulas, find the values of the expressions *without* explicitly calculating $x_1$ and $x_2$

   a. ${x_1}^2 + {x_2}^2 =$

   b. $(x_1 - x_2)^2 =$

   c. $\frac{1}{x_1} + \frac{1}{x_2} =$

   d. (*) ${x_1}^3 + {x_2}^3 =$

3. Solve the biquadratic equations: (a) $x^4 - 3x^2 + 2 = 0$ (b) $x^4 - x^2 - 2 = 0$.
4. What is the sum of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$? What is the product of those roots? Could you guess the roots?
5. Without solving the equation $3x^2 - 5x + 1 = 0$, find the arithmetic mean of its roots (that is $\frac{x_1+x_2}{2}$) and their geometric mean (that is $\sqrt{(x_1 x_2)}$ ).

6. Without solving the equation $x^2 - 12x + 19 = 0$ find the value of the following expression:
$$x_1(1 - x_1) + x_2(1 - x_2).$$

7. Find all numbers $\boldsymbol{p}$ such that sum of squares of the roots ($x_1^2 + x_2^2$) of the equation $x^2 - px + p + 7 = 0$ is equal to 10.

8. If $x_1, x_2$ are solutions for the quadratic equation $x^2 - 5x + p^2 - 2p + 1 = 0$, where $\boldsymbol{p}$ is some number, find the value of $\boldsymbol{p}$ so that the product of solutions of the equation is *minimal*.

9. Solve the equation $(x^2 + 2)^2 = 6x^2 + 4$. [Hint: Of course, you can just use the $(a + b)^2$ formula. Alternatively, one of the ways to solve it is to assume that $t = x^2 + 2$. Then the equation can be rewritten as a quadratic equation with $\boldsymbol{t}$ as a variable.]

10. (*) Find all numbers $\boldsymbol{p}$ and $\boldsymbol{q}$ such that the equation $x^2 + px + q = 0$ has solutions $\boldsymbol{p}$ and $\boldsymbol{q}$. {Hint: use Vieta's formulas]