

## Homework 11: Intro to quadratic equation

HW Due Date: January 18, 2026

### 1. Quadratic equation in a standard form.

Today we discussed how one solves quadratic equation, starting from the **standard form**:  $ax^2 + bx + c = 0$

A quadratic equation could have no solution, one solution, or two solutions depending on the coefficients a, b, and c.

We could solve such an equation by presenting it in a **factored form**:  $(x - x_1)(x - x_2) = 0$ , where  $x_1$  and  $x_2$  are the solutions of the equation, also known as *roots*. The factored form will also help us find a general formula for solving any quadratic equation using the coefficients a, b, c.

### 2. Solving the incomplete quadratic equation by factorizing.

➤ When  $c = 0$ ,  $ax^2 + bx = 0$

To solve, factorize as  $x(ax + b) = 0$  and the two terms in the product to be equal to zero. The two roots are  $x_1 = 0$  and  $x_2 = -b/a$

➤ When  $b = 0$ ,  $ax^2 + c = 0$

If  $c < 0$ , factorize the equation using the formula for fast multiplication  $a^2 - b^2 = (a - b)(a + b)$ . (\*)

For example,  $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0$ . Setting each term in the product to zero gives solutions of +5 and -5.

If  $c > 0$ , there are no real solutions. An easy way to see this is to solve directly for x:  $x^2 + 25 = 0 \Rightarrow x^2 = -5^2$ ; No number squared is equal to a negative number!

### 2. Solving the complete quadratic equation

➤ By completing the square

"Completing the square" works by using the formulas for fast multiplication  $(a \pm b)^2 = a^2 \pm 2ab + b^2$  (\*)

Here is an example how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x + 3)^2 - 7 = (x + 3)^2 - (\sqrt{7})^2 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

Thus,  $x^2 + 6x + 2 = 0$  if and only if  $(x + 3 + \sqrt{7}) = 0$ , which gives  $x = -3 - \sqrt{7}$ , or  $(x + 3 - \sqrt{7}) = 0$ , which gives  $x = -3 + \sqrt{7}$ .

➤ By using the quadratic formula

Completing the square works in general for any quadratic equation in a standard form

If  $a = 1$ , then:

$$x^2 + bx + c = x^2 + 2 \cdot \frac{b}{2}x + c = \left(x^2 + 2 \cdot \frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{2^2} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{2^2} \quad \text{eq (1)}$$

$$\text{Thus } x^2 + bx + c = 0 \text{ is equivalent to: } \left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$$

If  $a \neq 1$ , then:  $ax^2 + bx + c = 0$  is equivalent to:  $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$ , where  $D = b^2 - 4ac$

**The determinant** D determines the number of solutions.  $D < 0$ , there are no real solutions; if  $D = 0$ , there is one solution,

if  $D > 0$ , the solutions are:

$$x + \frac{b}{2a} = \pm \sqrt{\frac{D}{4a^2}}$$
$$x = \frac{-b \pm \sqrt{D}}{2a} \quad \text{eq (2)}$$

Homework problems

(\*) The parameters  $a$  and  $b$  in the formulas for fast multiplication  $a^2 - b^2 = (a - b)(a + b)$ ,  $(a \pm b)^2 = a^2 \pm 2ab + b^2$  are not the same as the coefficients a, b, and c used in the standard form of the quadratic equation!

**Instructions:** Please always write solutions on a **separate sheet of paper**. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

**Note:** Use the formulas for fast multiplication  $a^2 - b^2 = (a - b)(a + b)$ ,  $(a \pm b)^2 = a^2 \pm 2ab + b^2$ .

1. This problem requires that you carefully check your work and think:
  - a. Use formula (1) to prove that for any  $x$ ,  $x^2 + bx + c \geq -D/4$ , with equality only when  $x = -b/2$ .
  - b. Find the minimal possible value of the expression  $x^2 + 4x + 2$  [Hint: use part a) or complete the square]
  - c. Given a number  $a > 0$ , find the maximal possible value of the expression  $x(a - x)$  (the answer will depend on the value or values of  $a$ . In this case,  $a$  is called a *parameter*).
2. Derive Eq. 2 (given in previous page) for the most general quadratic equation  $ax^2 + bx + c = 0$ .
3. Convert the following equations to standard form (open brackets). Determine the coefficients  $a$ ,  $b$ , and  $c$ . Do not solve the equations!
  - a.  $2(x - 3)(x - 1) = 0$
  - b.  $(x - 2)^2 + (2x + 3)^2 = 13 - 4x$
4. Solve the following quadratic equations by converting to factorized form.
  - a.  $2x^2 - 3x = 0$
  - b.  $x^2 - 15 = 1$
  - c.  $3x^2 - 9 = 0$
  - d.  $2(x - 3)(x - 1) = 0$
5. Complete the square and find the solutions for the following quadratic equations:
  - a.  $x^2 + 4x + 3 = 0$
  - b.  $y^2 + 4y - 5 = 0$
6. Solve the following equations. Carefully think what method you will use and write all steps in your argument. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients  $a$ ,  $b$ ,  $c$ ? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (2)?
  - a.  $\frac{x}{x-2} = x - 1$
  - b.  $2x(3 - x) = 1$
  - c.  $x^3 + 4x^2 - 45x = 0$
7. If  $x + \frac{1}{x} = 7$ , find  $x^2 + \frac{1}{x^2} = 7$  and  $x^3 + \frac{1}{x^3}$  [Hint: try completing the square, completing the cube ...]
8. (\*) Consider the sequence  $x_1 = 1$ ,  $x_2 = \frac{x_1}{2} + \frac{1}{x_1}$ ,  $x_3 = \frac{x_2}{2} + \frac{1}{x_2}$  ...

Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you **guess** this value? [Hint: solve equation  $x = \frac{x}{2} + \frac{1}{x}$ ]

(\*) The parameters  $a$  and  $b$  in the formulas for fast multiplication  $a^2 - b^2 = (a - b)(a + b)$ ,  $(a \pm b)^2 = a^2 \pm 2ab + b^2$  are not the same as the coefficients  $a$ ,  $b$ , and  $c$  used in the standard form of the quadratic equation!