MATH 7: HANDOUT 24

TRIGONOMETRIC EQUATIONS AND INVERSE FUNCTIONS

Introduction

We have learned to evaluate $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Now we ask the *reverse* question: given a value, find the angle. Two problems motivate this:

- $\sin x = \frac{1}{2}$: the answer is in our table ($x = \frac{\pi}{6}$, plus others). Easy.
- $\sin x = 0.7$: the answer is *not* in our table. We need a new tool.

That tool is the **inverse trigonometric functions**.

Part I: Inverse Trigonometric Functions

The Arcsine Function

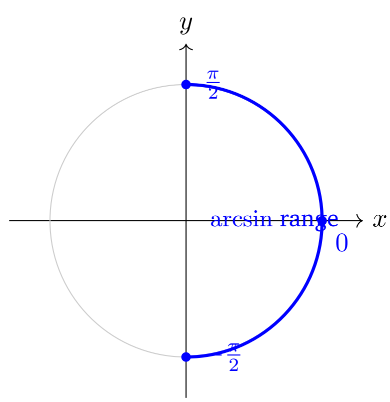
The sine function takes an angle and returns a number in $[-1, 1]$. The arcsine "undoes" this: it takes a number in $[-1, 1]$ and returns an angle. The catch is that infinitely many angles have the same sine, so we must choose a standard range.

Arcsine

The **arcsine** of y, written $\arcsin y$ (or $\sin^{-1} y$), is the unique angle θ satisfying

$$\sin\theta = y \qquad \text{and} \qquad -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}.$$

The range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the right half of the unit circle: every sine value in $[-1, 1]$ appears exactly once there.



Example 1. $\arcsin\dfrac{1}{2}$: need $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin\theta = \frac{1}{2}$. Answer: $\theta = \dfrac{\pi}{6}$.

Example 2. $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$: need $\sin\theta = -\frac{\sqrt{2}}{2}$ in the range. Since $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$ and $-\frac{\pi}{4}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$: answer $= -\dfrac{\pi}{4}$.

y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin y$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

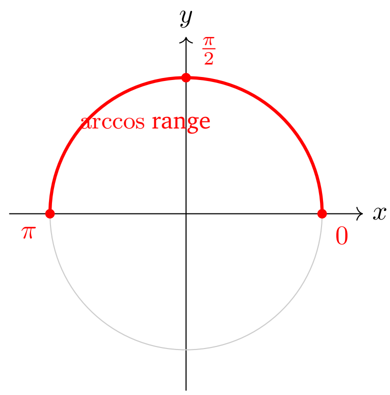
The Arccosine Function

Arccosine

The **arccosine** of y, written $\arccos y$ (or $\cos^{-1} y$), is the unique angle θ satisfying

$$\cos\theta = y \qquad \text{and} \qquad 0 \le \theta \le \pi.$$

The range $[0, \pi]$ is the upper half of the unit circle.



Example 3. $\arccos\frac{1}{2} = \frac{\pi}{3}$, since $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\frac{\pi}{3} \in [0, \pi]$.

Example 4. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, since $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\frac{5\pi}{6} \in [0, \pi]$.

y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arccos y$	π	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0

The Arctangent Function

Arctangent

The **arctangent** of y, written $\arctan y$ (or $\tan^{-1} y$), is the unique angle θ satisfying

$$\tan\theta = y \qquad \text{and} \qquad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}.$$

The range is an *open* interval because $\tan\theta$ is undefined at $\pm\frac{\pi}{2}$. Unlike arcsin and arccos, the domain of arctan is all real numbers — you can take $\arctan y$ for any y.

Example 5. $\arctan 1 = \frac{\pi}{4}, \quad \arctan(-\sqrt{3}) = -\frac{\pi}{3}, \quad \arctan 0 = 0.$

Quick Check

1. Find $\arcsin\frac{\sqrt{3}}{2}, \quad \arccos\frac{\sqrt{3}}{2}, \quad \arctan\sqrt{3}$. What do you notice?
2. Find $\arcsin(-1)$ and $\arccos(-1)$.
3. Why is $\arcsin(2)$ undefined?

Key Identity: $\arcsin y + \arccos y = \dfrac{\pi}{2}$

For any $y \in [-1, 1]$:

$$\boxed{\arcsin y + \arccos y = \frac{\pi}{2}.}$$

Why? Let $\theta = \arcsin y$, so $\sin\theta = y$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\cos(\frac{\pi}{2} - \theta) = \sin\theta = y$, and $\frac{\pi}{2} - \theta \in [0, \pi]$. So $\arccos y = \frac{\pi}{2} - \theta$, giving $\arcsin y + \arccos y = \theta + (\frac{\pi}{2} - \theta) = \frac{\pi}{2}$. □

Check: $\arcsin\frac{1}{2} + \arccos\frac{1}{2} = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$. ✓

Using a Calculator

For values not in the standard table (like $\sin x = 0.7$), use the **inverse trig keys** on a calculator.

- On most calculators: `sin⁻¹`, `cos⁻¹`, `tan⁻¹` (often accessed via **2nd** or **Shift** + the trig key).
- **Always check your mode** (degrees or radians) before calculating. In mathematics we usually work in radians.
- To convert: radians $\times \dfrac{180}{\pi}$ = degrees.

Example 6. Calculator examples (radian mode):

- $\arcsin(0.7) \approx 0.7754$ rad $\approx 44.4°$
- $\arccos(0.3) \approx 1.2661$ rad $\approx 72.5°$
- $\arctan(2) \approx 1.1071$ rad $\approx 63.4°$
- $\arcsin(-0.4) \approx -0.4115$ rad $\approx -23.6°$

Note that $\arcsin$ always gives a value in $[-90°, 90°]$ and $\arccos$ always gives a value in $[0°, 180°]$.

Quick Check

4. Use a calculator to find (in radians, to 3 decimal places): $\arcsin(0.5)$, $\arccos(0.5)$, $\arctan(1)$. Do the exact values match?
5. Use a calculator: $\arcsin(0.85)$, $\arccos(-0.6)$, $\arctan(5)$.

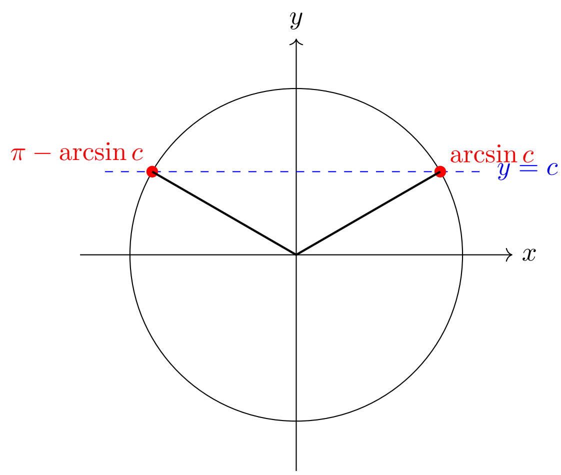
Part II: Trigonometric Equations

Now we can solve $\sin x = c$, $\cos x = c$, $\tan x = c$ for *any* value of c — not just values in our table.

Solving $\sin x = c$

From the unit circle: if $\sin \alpha = c$, then the point at angle $\pi - \alpha$ also has sine c (same y-coordinate). All other solutions are obtained by adding multiples of 2π.

$$\boxed{\sin x = c \iff x = \arcsin c + 2\pi n \quad \text{or} \quad x = \pi - \arcsin c + 2\pi n, \quad n \in \mathbb{Z}}$$



Example 7. Standard value. Solve $\sin x = \dfrac{1}{2}$.

$\arcsin \dfrac{1}{2} = \dfrac{\pi}{6}$, so:

$$x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad x = \pi - \frac{\pi}{6} + 2\pi n = \frac{5\pi}{6} + 2\pi n, \quad n \in \mathbb{Z}.$$

Example 8. Non-standard value. Solve $\sin x = 0.7$.

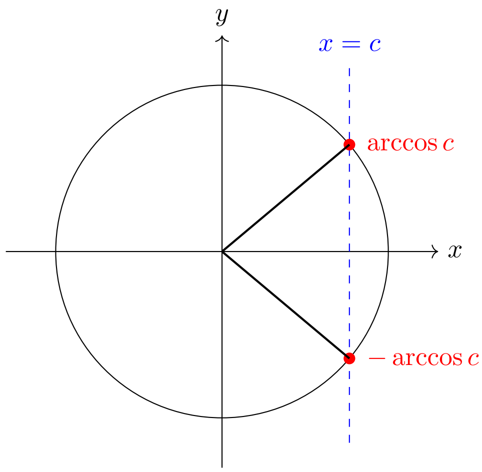
$\arcsin(0.7) \approx 0.7754$ rad. Using the formula:

$$x \approx 0.7754 + 2\pi n \quad \text{or} \quad x \approx \pi - 0.7754 + 2\pi n \approx 2.3662 + 2\pi n, \quad n \in \mathbb{Z}.$$

Solving $\cos x = c$

From the unit circle: if $\cos \alpha = c$, then $\cos(-\alpha) = c$ as well (same x-coordinate).

$$\boxed{\cos x = c \iff x = \pm \arccos c + 2\pi n, \quad n \in \mathbb{Z}}$$



Example 9. Standard value. Solve $\cos x = -\dfrac{1}{2}$.

$\arccos(-\frac{1}{2}) = \dfrac{2\pi}{3}$, so:

$$x = \pm\frac{2\pi}{3} + 2\pi n, \quad n \in \mathbb{Z}.$$

Example 10. Non-standard value. Solve $\cos x = 0.3$.

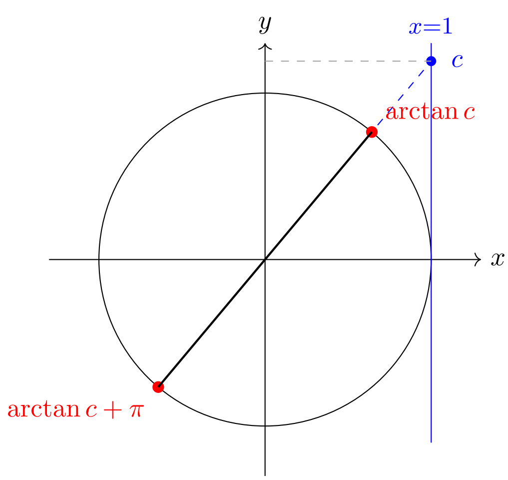
$\arccos(0.3) \approx 1.2661$ rad:

$$x \approx \pm 1.2661 + 2\pi n, \quad n \in \mathbb{Z}.$$

Solving $\tan x = c$

The tangent has period π, and for any c there is exactly one solution per period.

$$\boxed{\tan x = c \iff x = \arctan c + \pi n, \quad n \in \mathbb{Z}}$$



Example 11. Solve $\tan x = -\sqrt{3}$: $\arctan(-\sqrt{3}) = -\dfrac{\pi}{3}$, so $x = -\dfrac{\pi}{3} + \pi n$.

Example 12. Solve $\tan x = 2$ (calculator): $\arctan(2) \approx 1.1071$, so $x \approx 1.1071 + \pi n$.

Quick Check

6. Solve $\sin x = -\dfrac{\sqrt{3}}{2}$ and $\cos x = -\dfrac{\sqrt{2}}{2}$ (exact).
7. Solve $\sin x = 0.4$ and $\cos x = -0.7$ (use calculator, give decimal answers).
8. Solve $\tan x = 1$ and $\tan x = -3$ (the second one needs a calculator).

Equations with Transformed Arguments

When the argument is not just x, substitute u for the argument, solve for u, then recover x.

Example 13. Solve $\sin(2x) = \dfrac{1}{2}$.

Let $u = 2x$. Solve $\sin u = \frac{1}{2}$: $u = \dfrac{\pi}{6} + 2\pi n$ or $u = \dfrac{5\pi}{6} + 2\pi n$.
Divide by 2:

$$x = \frac{\pi}{12} + \pi n \quad \text{or} \quad x = \frac{5\pi}{12} + \pi n, \quad n \in \mathbb{Z}.$$

Example 14. Solve $\cos\left(x + \dfrac{\pi}{6}\right) = 0$.

Let $u = x + \dfrac{\pi}{6}$. Solve $\cos u = 0$: $u = \dfrac{\pi}{2} + \pi n$. Then $x = \dfrac{\pi}{2} - \dfrac{\pi}{6} + \pi n = \dfrac{\pi}{3} + \pi n$.

Quick Check

9. Solve $\sin(3x) = 0$.
10. Solve $\cos\left(x - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$.
11. Solve $\tan(2x) = 1$.

Quadratic Trigonometric Equations

Some equations become quadratic after substitution. Use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ to reduce everything to one function.

Example 15. Solve $2\sin^2 x - 3\sin x + 1 = 0$.

Let $y = \sin x$: $2y^2 - 3y + 1 = (2y-1)(y-1) = 0$, so $y = \frac{1}{2}$ or $y = 1$.

- $\sin x = \frac{1}{2}$: $x = \frac{\pi}{6} + 2\pi n$ or $x = \frac{5\pi}{6} + 2\pi n$.
- $\sin x = 1$: $x = \frac{\pi}{2} + 2\pi n$.

Example 16. Solve $\sin^2 x + 3\cos^2 x = 2$.

Replace $\cos^2 x = 1 - \sin^2 x$: $\sin^2 x + 3(1 - \sin^2 x) = 2$, giving $3 - 2\sin^2 x = 2$, so $\sin^2 x = \frac{1}{2}$, $\sin x = \pm\dfrac{\sqrt{2}}{2}$.

$$x = \pm\frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}.$$

Quick Check

12. Solve $\cos^2 x - \cos x = 0$.
13. Solve $2\cos^2 x - \cos x - 1 = 0$.

Key Takeaways

- $\arcsin y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos y \in [0, \pi]$, $\arctan y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\arcsin y + \arccos y = \frac{\pi}{2}$ for all $y \in [-1, 1]$.
- Calculator keys: `sin⁻¹`, `cos⁻¹`, `tan⁻¹`. Always check radian/degree mode.
- $\sin x = c$: two families, $\arcsin c + 2\pi n$ and $\pi - \arcsin c + 2\pi n$.
- $\cos x = c$: one family, $\pm \arccos c + 2\pi n$.
- $\tan x = c$: one family, $\arctan c + \pi n$.
- For $\sin(ax + b) = c$: substitute $u = ax + b$, solve, divide back.
- For quadratics: substitute $y = \sin x$ (or $\cos x$), solve, check $|y| \leq 1$.

Common Mistakes

- $\sin^{-1} x \neq \frac{1}{\sin x}$. The -1 is a function inverse, not a power. $\frac{1}{\sin x} = \csc x$.
- **Wrong range for arccos.** $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$, not $-\frac{\pi}{3}$ (which is outside $[0, \pi]$).
- **Missing the second family.** $\sin x = c$ has *two* families per 2π; $\cos x = c$ also has two (the $\pm$).
- **Calculator mode.** If your calculator is in degree mode, $\arcsin(0.7) \approx 44.4$ not 0.775.
- **Forgetting $n \in \mathbb{Z}$.** Always write the general solution with $+2\pi n$ or $+\pi n$.
- **Discarding invalid roots.** If a quadratic gives $\sin x = 1.5$, that branch has no solutions.

Classwork

1. Evaluate without a calculator:
   (a) $\arcsin\dfrac{\sqrt{3}}{2}$ (b) $\arccos\left(-\dfrac{1}{2}\right)$ (c) $\arctan(-1)$ (d) $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$
2. Use a calculator (radian mode, 3 decimal places):
   (a) Find $\arcsin(0.45)$ and $\arcsin(-0.45)$. What do you notice?
   (b) Find $\arccos(-0.8)$, then write the general solution of $\cos x = -0.8$.
3. Solve exactly. Write all solutions with $n \in \mathbb{Z}$.
   (a) $\sin x = -\dfrac{1}{2}$ (c) $\tan x = \sqrt{3}$ (e) $\tan x = -\dfrac{1}{\sqrt{3}}$
   (b) $\cos x = \dfrac{\sqrt{3}}{2}$ (d) $\cos x = -\dfrac{\sqrt{2}}{2}$
4. Simplify without a calculator:
   (a) $\sin(\arcsin 0.5)$ (b) $\cos(\arccos(-0.5))$ (c) $\arcsin\left(\sin\dfrac{\pi}{6}\right)$
5. Solve $\cos\left(x - \dfrac{\pi}{3}\right) = -\dfrac{1}{2}$.
6. Solve:
   (a) $2\sin^2 x + \sin x - 1 = 0$ (b) $2\cos^2 x + 3\cos x + 1 = 0$

Classwork Solutions

1. (a) $\arcsin\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}$
   (b) $\arccos\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$
   (c) $\arctan(-1) = -\dfrac{\pi}{4}$
   (d) $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{\pi}{4}$
2. (a) $\arcsin(0.45) \approx 0.467$; $\arcsin(-0.45) \approx -0.467$. They are negatives of each other — arcsin is an odd function.
   (b) $\arccos(-0.8) \approx 2.498$. General solution: $x \approx \pm 2.498 + 2\pi n$, $n \in \mathbb{Z}$.
3. (a) $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, so $x = -\dfrac{\pi}{6} + 2\pi n$ or $x = \dfrac{7\pi}{6} + 2\pi n$.
   (b) $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, so $x = \pm\dfrac{\pi}{6} + 2\pi n$.
   (c) $\arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$, so $x = \dfrac{\pi}{3} + \pi n$.
   (d) $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$, so $x = \pm\dfrac{3\pi}{4} + 2\pi n$.
   (e) $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$, so $x = -\dfrac{\pi}{6} + \pi n$.
4. (a) $\sin(\arcsin 0.5) = 0.5$
   (b) $\cos(\arccos(-0.5)) = -0.5$
   (c) $\arcsin\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$ ($\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so no adjustment needed)

5. Let $u = x - \frac{\pi}{3}$. Solve $\cos u = -\frac{1}{2}$: $u = \pm\dfrac{2\pi}{3} + 2\pi n$.

$$x = \frac{\pi}{3} + \frac{2\pi}{3} + 2\pi n = \pi + 2\pi n \qquad \text{or} \qquad x = \frac{\pi}{3} - \frac{2\pi}{3} + 2\pi n = -\frac{\pi}{3} + 2\pi n.$$

6. (a) Let $y = \sin x$: $(2y-1)(y+1) = 0$, so $y = \frac{1}{2}$ or $y = -1$.
$\sin x = \frac{1}{2}$: $x = \frac{\pi}{6} + 2\pi n$ or $x = \frac{5\pi}{6} + 2\pi n$. $\sin x = -1$: $x = -\frac{\pi}{2} + 2\pi n$.

   (b) Let $y = \cos x$: $(2y+1)(y+1) = 0$, so $y = -\frac{1}{2}$ or $y = -1$.
$\cos x = -\frac{1}{2}$: $x = \pm\frac{2\pi}{3} + 2\pi n$. $\cos x = -1$: $x = \pi + 2\pi n$.

Homework

1. Evaluate without a calculator:
   (a) $\arcsin 1$
   (b) $\arccos(-1)$
   (c) $\arctan(-1)$
   (d) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
   (e) $\arccos\left(\frac{\sqrt{2}}{2}\right)$
   (f) $\arctan\left(-\frac{1}{\sqrt{3}}\right)$

2. Use a calculator (radian mode) to find each value to 3 decimal places. Then write the general solution of the corresponding equation.
   (a) $\arcsin(0.6)$; hence solve $\sin x = 0.6$.
   (b) $\arccos(-0.4)$; hence solve $\cos x = -0.4$.
   (c) $\arctan(3)$; hence solve $\tan x = 3$.

3. Solve exactly. Write all solutions with $n \in \mathbb{Z}$.
   (a) $\sin x = \sin \dfrac{\pi}{5}$
   (b) $\cos\left(x + \dfrac{\pi}{6}\right) = 0$

4. Simplify (no calculator). Think carefully about the range before answering (b) and (c).
   (a) $\arccos\left(\cos \dfrac{2\pi}{3}\right)$
   (b) $\arcsin\left(\sin \dfrac{3\pi}{4}\right)$
   (c) $\arccos\left(\cos \dfrac{7\pi}{4}\right)$

5. Solve the following equations:
   (a) $\sin(3x) = \dfrac{1}{2}$
   (b) $\cos(2x) = -\dfrac{\sqrt{3}}{2}$
   (c) $\tan\left(x + \dfrac{\pi}{4}\right) = 0$

6. Solve the following equations:
   (a) $4\cos^2 x - 3 = 0$
   (b) $\sin^2 x + 3\cos^2 x = 2$

7. Find the exact value of each expression. *Hint: if* $\theta = \arccos \frac{3}{5}$, *then* $\cos\theta = \frac{3}{5}$*; draw a right triangle with adjacent* $= 3$, *hypotenuse* $= 5$, *and find the missing side using the Pythagorean theorem.*
   (a) $\sin\left(\arccos \dfrac{3}{5}\right)$
   (b) $\cos\left(\arcsin \dfrac{5}{13}\right)$
   (c) $\tan\left(\arcsin \dfrac{4}{5}\right)$

8. For which $x \in [-1, 1]$ is $\arcsin x > \arccos x$? Justify your answer.
   Hint: use the identity $\arcsin x + \arccos x = \frac{\pi}{2}$.

9. **M** Without a calculator, show that $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$ for all $x > 0$.
   Hint: let $\theta = \arctan x$, *so* $\tan\theta = x$. *Use the reduction formula* $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\theta}$ *from Handout 23 to find* $\arctan \dfrac{1}{x}$.

10. **M** Without a calculator, determine which is larger: $\arcsin \dfrac{2}{3}$ or $\arctan \dfrac{2}{3}$? Justify your answer.
    Hint: draw right triangles for each, then compare them by computing $\tan$ *of both angles.*

11. **H** Solve $\arcsin x = \arccos(2x)$.
    Hint: apply $\sin$ *to both sides. To compute* $\sin(\arccos(2x))$, *draw a right triangle with adjacent leg* $2x$ *and hypotenuse* 1.

Quick Check Answers

Arcsine, Arccosine, Arctangent (QC 1–3)

1. $\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$; $\arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$; $\arctan\sqrt{3} = \frac{\pi}{3}$. Notice $\arcsin + \arccos = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$.
2. $\arcsin(-1) = -\frac{\pi}{2}$; $\quad \arccos(-1) = \pi$.
3. The domain of arcsin is $[-1, 1]$, and $2 > 1$, so $\arcsin(2)$ is undefined.

Calculator (QC 4–5)

4. $\arcsin(0.5) \approx 0.524 = \frac{\pi}{6}$; $\arccos(0.5) \approx 1.047 = \frac{\pi}{3}$; $\arctan(1) \approx 0.785 = \frac{\pi}{4}$. Yes, the decimal values match the exact answers.
5. $\arcsin(0.85) \approx 1.016$; $\quad \arccos(-0.6) \approx 2.214$; $\quad \arctan(5) \approx 1.373$.

Trigonometric Equations (QC 6–8)

6. $\sin x = -\frac{\sqrt{3}}{2}$: $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$, so $x = -\frac{\pi}{3} + 2\pi n$ or $x = \frac{4\pi}{3} + 2\pi n$.
   $\cos x = -\frac{\sqrt{2}}{2}$: $\arccos(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$, so $x = \pm\frac{3\pi}{4} + 2\pi n$.
7. $\sin x = 0.4$: $\arcsin(0.4) \approx 0.412$, so $x \approx 0.412 + 2\pi n$ or $x \approx 2.730 + 2\pi n$.
   $\cos x = -0.7$: $\arccos(-0.7) \approx 2.346$, so $x \approx \pm 2.346 + 2\pi n$.
8. $\tan x = 1$: $x = \frac{\pi}{4} + \pi n$. $\quad \tan x = -3$: $\arctan(-3) \approx -1.249$, so $x \approx -1.249 + \pi n$.

Transformed Arguments (QC 9–11)

9. $\sin(3x) = 0$: $3x = \pi n$, so $x = \dfrac{\pi n}{3}$.
10. $\cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$: $x - \frac{\pi}{4} = \pm\frac{\pi}{4} + 2\pi n$, giving $x = \frac{\pi}{2} + 2\pi n$ or $x = 2\pi n$.
11. $\tan(2x) = 1$: $2x = \frac{\pi}{4} + \pi n$, so $x = \frac{\pi}{8} + \frac{\pi n}{2}$.

Quadratic Equations (QC 12–13)

12. $\cos^2 x - \cos x = 0$: $\cos x(\cos x - 1) = 0$, so $\cos x = 0$ or $\cos x = 1$.
    $\cos x = 0$: $x = \frac{\pi}{2} + \pi n$. $\quad \cos x = 1$: $x = 2\pi n$.
13. $2\cos^2 x - \cos x - 1 = 0$: $(2\cos x + 1)(\cos x - 1) = 0$, so $\cos x = -\frac{1}{2}$ or $\cos x = 1$.
    $\cos x = -\frac{1}{2}$: $x = \pm\frac{2\pi}{3} + 2\pi n$. $\quad \cos x = 1$: $x = 2\pi n$.