MATH 7: HANDOUT 24

TRIGONOMETRY V: INVERSE FUNCTIONS AND TRIGONOMETRIC EQUATIONS (SUMMARY)

Inverse Trigonometric Functions

Many angles can have the same sine, so to make $\arcsin y$ unambiguous we restrict the output angle to a standard interval (the *principal interval*). Similarly for $\arccos$ and $\arctan$.

Function	**Allowed input** y	**Output angle** θ
$\arcsin y$	$y \in [-1, 1]$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos y$	$y \in [-1, 1]$	$\theta \in [0, \pi]$
$\arctan y$	$y \in \mathbb{R}$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\arcsin y$ is the unique angle θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin\theta = y$ (similarly for $\arccos$ and $\arctan$).

Key identity. For all $y \in [-1, 1]$:

$$\boxed{\arcsin y + \arccos y = \frac{\pi}{2}}$$

Standard Values

y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin y$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\arccos y$	π	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0

Using a Calculator

Inverse trig keys: `sin⁻¹`, `cos⁻¹`, `tan⁻¹` (often via **2nd/Shift**). **Always check radian vs. degree mode.** For non-standard values, e.g. $\arcsin(0.7) \approx 0.7754$ rad $\approx 44.4°$.

When Compositions Don't Cancel

$\arcsin(\sin x) = x$ *only* when $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Otherwise reduce first:

$$\arcsin\left(\sin \tfrac{3\pi}{4}\right) = \arcsin \tfrac{\sqrt{2}}{2} = \tfrac{\pi}{4} \quad \text{(not } \tfrac{3\pi}{4}\text{)}.$$

Similarly $\arccos(\cos x) = x$ only when $x \in [0, \pi]$.

Trigonometric Equations

For any value c for which the right-hand side is defined, the general solution is:

$$\boxed{\sin x = c \iff x = \arcsin c + 2\pi n \text{ or } x = \pi - \arcsin c + 2\pi n}$$

$$\boxed{\cos x = c \iff x = \pm\arccos c + 2\pi n} \qquad \boxed{\tan x = c \iff x = \arctan c + \pi n}$$

($n \in \mathbb{Z}$ in all cases.) For $\sin$ and $\cos$, the equation has *no* solutions when $|c| > 1$.

Transformed Arguments

For $\sin(ax + b) = c$ (or $\cos$, $\tan$): substitute $u = ax + b$, solve for u, then recover $x = (u - b)/a$.

Quadratic Trig Equations

For equations involving $\sin^2 x$ or $\cos^2 x$: substitute $y = \sin x$ (or $\cos x$) to get a quadratic in y. Use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ to reduce to a single function if needed. Always check $|y| \leq 1$ before solving each branch.

Key Takeaways

- $\arcsin y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad \arccos y \in [0, \pi], \quad \arctan y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\arcsin y + \arccos y = \frac{\pi}{2}$ for all $y \in [-1, 1]$.
- $\sin x = c$: two families, $\arcsin c + 2\pi n$ and $\pi - \arcsin c + 2\pi n$.
- $\cos x = c$: one family, $\pm \arccos c + 2\pi n$.
- $\tan x = c$: one family, $\arctan c + \pi n$.
- For $\sin(ax + b) = c$: substitute $u = ax + b$, solve, then divide back.
- For quadratics: substitute $y = \sin x$ (or $\cos x$); discard branches with $|y| > 1$.

Common Mistakes

- $\sin^{-1} x$ **is the inverse function, not a reciprocal.** $\dfrac{1}{\sin x} = \csc x$.
- **Wrong principal value for arccos.** $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$, not $-\frac{\pi}{3}$.
- **Missing the second family.** $\sin x = c$ has *two* families per 2π; $\cos x = c$ also has two (the $\pm$).
- **Calculator mode.** Mixing degrees and radians gives wildly wrong answers.
- **Forgetting** $n \in \mathbb{Z}$ in the general solution.
- **Discarding invalid roots.** If a quadratic gives $\sin x = 1.5$, that branch has no solutions.
- **Blindly cancelling.** $\arcsin(\sin x) = x$ *only* when $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Homework

1. Evaluate without a calculator:
   (a) $\arcsin 1$
   (b) $\arccos(-1)$
   (c) $\arctan(-1)$
   (d) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
   (e) $\arccos\left(\frac{\sqrt{2}}{2}\right)$
   (f) $\arctan\left(-\frac{1}{\sqrt{3}}\right)$

2. Use a calculator (radian mode) to find each value to 3 decimal places. Then write the general solution of the corresponding equation.
   (a) $\arcsin(0.6)$; hence solve $\sin x = 0.6$.
   (b) $\arccos(-0.4)$; hence solve $\cos x = -0.4$.
   (c) $\arctan(3)$; hence solve $\tan x = 3$.

3. Solve exactly. Write all solutions with $n \in \mathbb{Z}$.
   (a) $\sin x = \sin \dfrac{\pi}{5}$
   (b) $\cos\left(x + \dfrac{\pi}{6}\right) = 0$

4. Simplify (no calculator). Check the principal interval carefully before answering (b) and (c).
   (a) $\arccos\left(\cos \dfrac{2\pi}{3}\right)$
   (b) $\arcsin\left(\sin \dfrac{3\pi}{4}\right)$
   (c) $\arccos\left(\cos \dfrac{7\pi}{4}\right)$

5. Solve the following equations:
   (a) $\sin(3x) = \dfrac{1}{2}$
   (b) $\cos(2x) = -\dfrac{\sqrt{3}}{2}$
   (c) $\tan\left(x + \dfrac{\pi}{4}\right) = 0$

6. Solve the following equations:
   (a) $4\cos^2 x - 3 = 0$
   (b) $\sin^2 x + 3\cos^2 x = 2$

7. Find the exact value of each expression. *Hint: if $\theta = \arccos \frac{3}{5}$, then $\cos\theta = \frac{3}{5}$; draw a right triangle with adjacent $= 3$, hypotenuse $= 5$, and find the missing side using the Pythagorean theorem.*
   (a) $\sin\left(\arccos \dfrac{3}{5}\right)$
   (b) $\cos\left(\arcsin \dfrac{5}{13}\right)$
   (c) $\tan\left(\arcsin \dfrac{4}{5}\right)$

8. For which $x \in [-1, 1]$ is $\arcsin x > \arccos x$? Justify your answer.
   Hint: use the identity $\arcsin x + \arccos x = \frac{\pi}{2}$.

9. **M** Without a calculator, show that $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$ for all $x > 0$.
   Hint: let $\theta = \arctan x$, so $\tan\theta = x$. Use the reduction formula $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\theta}$ from Handout 23 to find $\arctan \dfrac{1}{x}$.

10. **M** Without a calculator, determine which is larger: $\arcsin \dfrac{2}{3}$ or $\arctan \dfrac{2}{3}$? Justify your answer.
    Hint: draw right triangles for each, then compare them by computing $\tan$ of both angles.

11. **H** Solve $\arcsin x = \arccos(2x)$.
    Hint: apply $\sin$ to both sides. To compute $\sin(\arccos(2x))$, draw a right triangle with adjacent leg $2x$ and hypotenuse 1.