MATH 7: HANDOUT 23

Trigonometry IV: Tangent, Identities, and Reduction Formulas

Introduction

A **trigonometric identity** is an equation involving trigonometric functions that is true for *all* values of the variable (where defined). We have already seen the most important one:

$$\sin^2\alpha + \cos^2\alpha = 1.$$

In this handout we first explore the tangent function through the unit circle and its graph, then develop the key identities and *reduction formulas* that allow us to simplify expressions like $\sin(\alpha + \frac{\pi}{2})$ or $\cos(\pi - \alpha)$.

How Computers Compute Sine and Cosine

Modern computers do *not* draw triangles, circles, or measure angles the way we do. Instead, they evaluate $\sin x$, $\cos x$, and $\tan x$ using clever algebraic methods that are extremely fast and accurate.

1. Taylor Series (Power Series). For angles measured in radians, sine and cosine can be written as infinite polynomials:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

A computer takes just the first few terms, which already give extremely accurate values when x is small.

2. Range Reduction. If x is large, the computer first reduces it to a small equivalent angle via $\sin(x) = \sin(x - 2\pi k)$, then evaluates the polynomial.

3. CORDIC Algorithm (1959). Some processors compute trig functions using only additions, subtractions, and bit shifts — no multiplication required. This made early pocket calculators possible.

In short: although trigonometry began with geometry, modern devices compute these values purely algebraically.

Tangent in the Trigonometric Circle

In previous handouts we defined $\tan\alpha = \sin\alpha/\cos\alpha$ algebraically. There is also a beautiful geometric picture: the tangent value can be read directly off the unit circle using a vertical line. This construction explains in one glance why $\tan$ is undefined at $\pm\frac{\pi}{2}$, why it changes sign between quadrants, and why it grows without bound near the asymptotes.

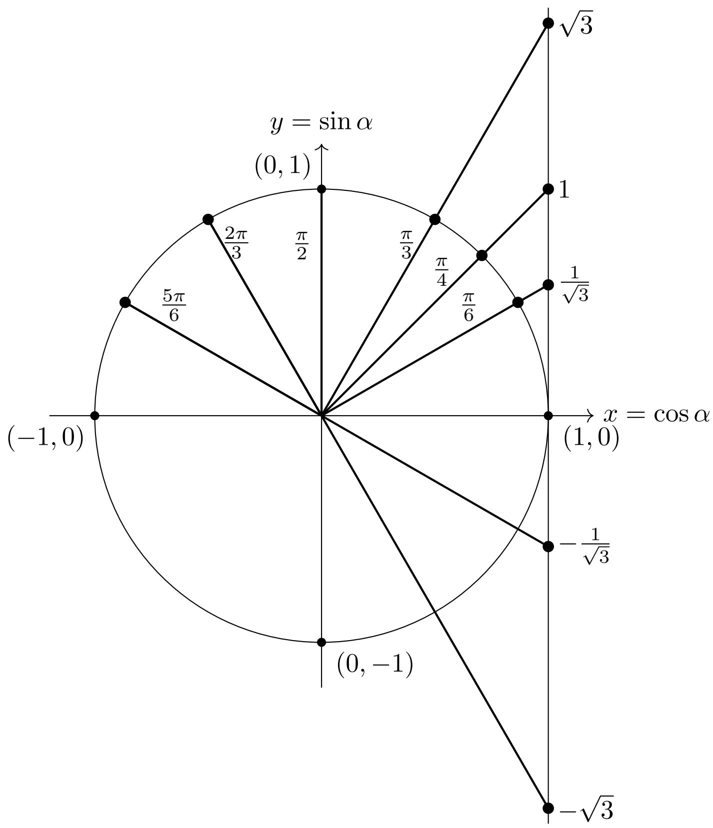
We understand the tangent function using the unit circle. Start with the point on the circle corresponding to the angle α, which has coordinates $(\cos\alpha,\ \sin\alpha)$. Draw the line that passes through the origin and this point. Next, draw the *vertical line* tangent to the unit circle at the point $(1, 0)$; this is the line $x = 1$.

The value of $\tan\alpha$ is defined as the y-coordinate of the intersection of these two lines. In other words, we extend the radius through angle α until it meets the vertical tangent line, and then read off the height of that intersection point.

This construction agrees with the familiar formula

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

whenever $0 < \alpha < \frac{\pi}{2}$, and it also allows us to define $\tan\alpha$ for any angle α for which $\cos\alpha \neq 0$. When $\cos\alpha = 0$, the line does not intersect the tangent, so the tangent function is undefined — consistent with the vertical asymptotes in the graph of $\tan x$.



Quick Check

1. Explain in words why $\tan\alpha$ is undefined when $\cos\alpha = 0$.
2. In which quadrants is $\tan x$ positive? In which quadrants is it negative?
3. Using the definition $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, compute:

$$\tan\left(\frac{\pi}{6}\right), \quad \tan\left(\frac{\pi}{4}\right), \quad \tan\left(\frac{\pi}{3}\right).$$

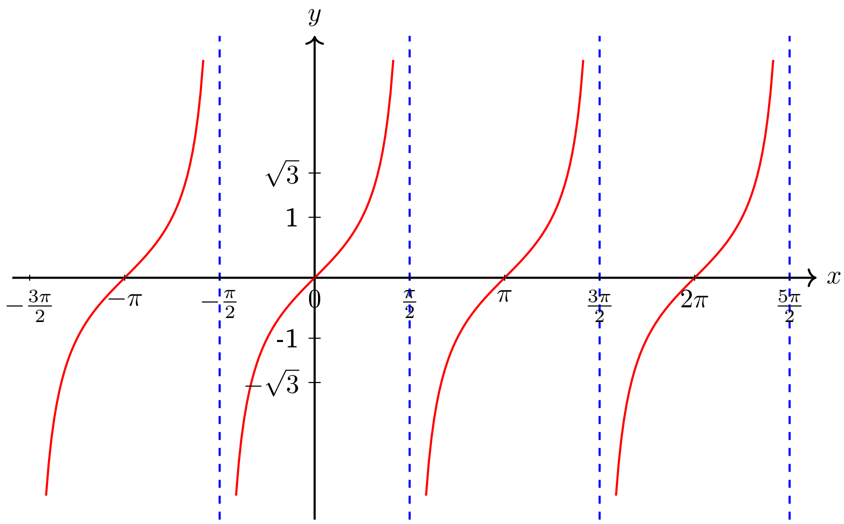
Graph of the Tangent Function

By looking at the values of $\tan x$ as we move around the trigonometric circle, several important facts become clear:

- $\tan 0 = 0$ and $\tan\pi = 0$.
- On the interval $0 < x < \frac{\pi}{2}$, the function $\tan x$ increases from 0 upward.
- As $x \to \frac{\pi}{2}^-$ (from the left), the value $\tan x$ grows without bound ("goes to infinity").
- As $x \to -\frac{\pi}{2}^+$ (from the right), the value $\tan x$ decreases without bound ("goes to negative infinity").
- The tangent function is periodic with period π:

$$\tan(x+\pi) = \tan x.$$

All of these features are visible in the graph of $y = \tan x$ below:



Quick Check

1. From the graph, find all x in $(-2\pi, 2\pi)$ where $\tan x = 0$.
2. From the graph, find all x in $(-2\pi, 2\pi)$ where $\tan x$ is undefined.
3. What is the smallest positive period of $\tan x$? Explain how you can see this from the graph.

The Pythagorean Identity Revisited

You have been using $\sin^2\alpha + \cos^2\alpha = 1$ since we first introduced the unit circle. It is worth pausing to appreciate how much work this single equation can do: given just one trigonometric value and the quadrant, it determines all the others. Let us practice this.

Recall the fundamental identity:

$$\boxed{\sin^2\alpha + \cos^2\alpha = 1}$$

From this, we can derive two useful consequences:

$$\sin^2\alpha = 1 - \cos^2\alpha$$
$$\cos^2\alpha = 1 - \sin^2\alpha$$

Example 1. If $\sin\alpha = \frac{3}{5}$ and α is in the first quadrant, find $\cos\alpha$ and $\tan\alpha$.

Using $\cos^2\alpha = 1 - \sin^2\alpha$:

$$\cos^2\alpha = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}.$$

Since α is in the first quadrant, $\cos\alpha > 0$, so $\cos\alpha = \frac{4}{5}$.

Then:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{3/5}{4/5} = \frac{3}{4}.$$

Example 2. If $\sin\alpha = -\frac{3}{5}$ and α is in the fourth quadrant, find $\cos\alpha$ and $\tan\alpha$.

Using the same calculation, $\cos^2\alpha = \frac{16}{25}$.

In the fourth quadrant, $\cos\alpha > 0$ but $\sin\alpha < 0$, so:

$$\cos\alpha = \frac{4}{5}, \qquad \tan\alpha = \frac{-3/5}{4/5} = -\frac{3}{4}.$$

Quick Check

1. If $\cos\alpha = \frac{5}{13}$ and α is in the first quadrant, find $\sin\alpha$.
2. If $\sin\alpha = -\frac{1}{2}$ and α is in the third quadrant, find $\cos\alpha$.
3. If $\tan\alpha = 2$ and α is in the first quadrant, find $\sin\alpha$ and $\cos\alpha$.

Related Identities for Tangent

Dividing the Pythagorean identity through by $\cos^2\alpha$ or $\sin^2\alpha$ yields two further identities. Along the way we formally introduce the **cotangent** — the reciprocal of tangent.

The **cotangent** is defined as

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\tan\alpha} \qquad (\sin\alpha \neq 0).$$

Dividing the Pythagorean identity by $\cos^2\alpha$ or $\sin^2\alpha$ gives two useful identities:

$$\boxed{1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}} \qquad \boxed{1 + \cot^2\alpha = \frac{1}{\sin^2\alpha}}$$

Example 3. If $\tan\alpha = \frac{1}{2}$ and α is in the first quadrant, find $\cos\alpha$ and $\sin\alpha$.

From $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$:

$$1 + \frac{1}{4} = \frac{1}{\cos^2\alpha} \quad \Rightarrow \quad \cos^2\alpha = \frac{4}{5} \quad \Rightarrow \quad \cos\alpha = \frac{2}{\sqrt{5}}.$$

Then:

$$\sin\alpha = \tan\alpha \cdot \cos\alpha = \frac{1}{2} \cdot \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}.$$

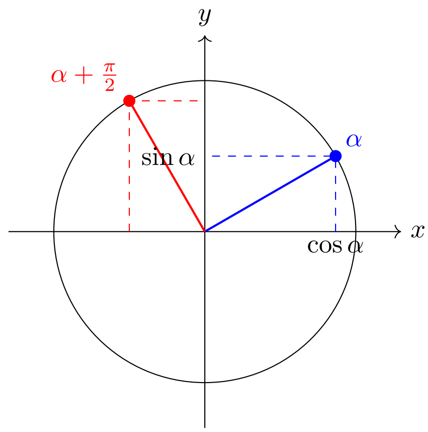
Reduction Formulas

The **reduction formulas** tell us how to simplify expressions like $\sin(\alpha + \frac{\pi}{2})$, $\cos(\pi - \alpha)$, and so on. These come up constantly in equation solving and simplification. The underlying idea is that the unit circle has symmetry: rotating or reflecting the point $(\cos\alpha, \sin\alpha)$ by a multiple of $\frac{\pi}{2}$ lands on another point whose coordinates are simple functions of $\cos\alpha$ and $\sin\alpha$ — possibly with the two functions swapped, possibly with a sign change.

Adding $\frac{\pi}{2}$ (or $90°$)

On the unit circle, adding $\frac{\pi}{2}$ to an angle rotates the point by $90°$ counterclockwise. This swaps the x and y coordinates (with a sign change):

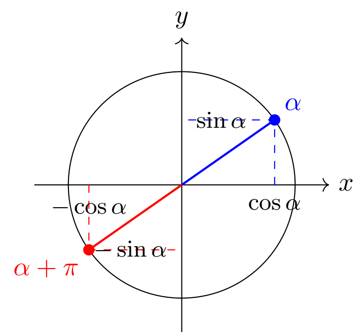
$$\boxed{\begin{aligned} \sin\left(\alpha + \frac{\pi}{2}\right) &= \cos\alpha \\ \cos\left(\alpha + \frac{\pi}{2}\right) &= -\sin\alpha \end{aligned}}$$



Adding π (or $180°$)

Adding π moves to the opposite point on the circle, negating both coordinates:

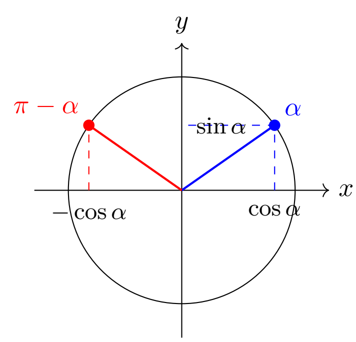
$$\boxed{\begin{aligned}\sin(\alpha+\pi) &= -\sin\alpha\\ \cos(\alpha+\pi) &= -\cos\alpha\end{aligned}}$$



Subtracting from π

The angle $\pi - \alpha$ reflects the point across the y-axis:

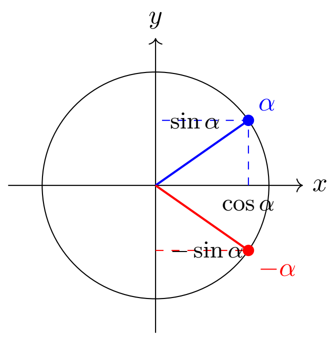
$$\boxed{\begin{aligned}\sin(\pi-\alpha) &= \sin\alpha\\ \cos(\pi-\alpha) &= -\cos\alpha\end{aligned}}$$



Negative Angles

The angle $-\alpha$ reflects across the x-axis:

$$\boxed{\begin{aligned}\sin(-\alpha) &= -\sin\alpha\\ \cos(-\alpha) &= \cos\alpha\end{aligned}}$$



Subtracting from $\frac{\pi}{2}$ (Cofunction Identities)

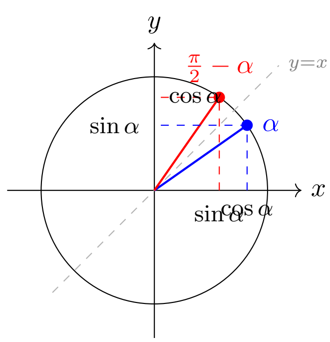
The angle $\frac{\pi}{2} - \alpha$ is the *complement* of α. Using the $+\frac{\pi}{2}$ rule and the negative-angle rule:

$$\sin\left(\tfrac{\pi}{2} - \alpha\right) = \sin\left(\tfrac{\pi}{2} + (-\alpha)\right) = \cos(-\alpha) = \cos\alpha,$$

$$\cos\left(\tfrac{\pi}{2} - \alpha\right) = \cos\left(\tfrac{\pi}{2} + (-\alpha)\right) = -\sin(-\alpha) = \sin\alpha.$$

$$\boxed{\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos\alpha\\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin\alpha\end{aligned}}$$

These are called the **cofunction identities**: sine and cosine of complementary angles are equal.



General Rule for Reduction

For any reduction formula, follow these steps:

1. **Does the function change?**
   - If the shift is an *odd* multiple of $\frac{\pi}{2}$ (like $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$), the function changes: $\sin \leftrightarrow \cos$, $\tan \leftrightarrow \cot$.
   - If the shift is an *even* multiple of $\frac{\pi}{2}$ (like $\pm\pi$, $\pm 2\pi$), the function stays the same.
2. **What is the sign?**
   - Assume α is a small acute angle (in the first quadrant).

- Determine which quadrant the new angle falls into.
- The sign matches the sign of the *original* function in that quadrant.

Example 4. Find $\tan\left(\frac{3\pi}{2}+\alpha\right)$.
Step 1: The shift is $\frac{3\pi}{2}$, which is an odd multiple of $\frac{\pi}{2}$, so $\tan \to \cot$.
Step 2: For small acute α, the angle $\frac{3\pi}{2}+\alpha$ is in the fourth quadrant, where $\tan$ is negative.
Therefore:

$$\tan\left(\frac{3\pi}{2}+\alpha\right) = -\cot\alpha.$$

Quick Check

1. Simplify $\cos\left(x-\frac{\pi}{2}\right)$.
2. Simplify $\sin(\pi+x)$.
3. Simplify $\tan(2\pi-x)$.
4. Simplify $\cos\left(x+\frac{3\pi}{2}\right)$.

Summary of Common Reduction Formulas

Angle	sin	cos	tan
$-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$
$\frac{\pi}{2}-\alpha$	$\cos\alpha$	$\sin\alpha$	$\cot\alpha$
$\frac{\pi}{2}+\alpha$	$\cos\alpha$	$-\sin\alpha$	$-\cot\alpha$
$\pi-\alpha$	$\sin\alpha$	$-\cos\alpha$	$-\tan\alpha$
$\pi+\alpha$	$-\sin\alpha$	$-\cos\alpha$	$\tan\alpha$
$\frac{3\pi}{2}-\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\cot\alpha$
$\frac{3\pi}{2}+\alpha$	$-\cos\alpha$	$\sin\alpha$	$-\cot\alpha$
$2\pi-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$

Using Identities to Simplify Expressions

Now we put everything together. The Pythagorean identity and the reduction formulas can be chained: first reduce a shifted argument, then apply $\sin^2\alpha+\cos^2\alpha=1$. Even expressions that look complicated often collapse to something very simple.

Example 5. Simplify $\sin^2 x+\sin^2\left(x+\frac{\pi}{2}\right)$.
Using the reduction formula $\sin\left(x+\frac{\pi}{2}\right)=\cos x$:

$$\sin^2 x+\sin^2\left(x+\frac{\pi}{2}\right)=\sin^2 x+\cos^2 x=1.$$

Example 6. If $\sin x=\frac{5}{13}$ and $0<x<\frac{\pi}{2}$, find $\cos\left(x+\frac{\pi}{2}\right)$.
Using the reduction formula:

$$\cos\left(x+\frac{\pi}{2}\right)=-\sin x=-\frac{5}{13}.$$

Key Takeaways

- **Pythagorean identity:** $\sin^2\alpha + \cos^2\alpha = 1$ allows you to find one function from the other (up to sign).
- **Tangent/cotangent identities:** $1 + \tan^2\alpha = \dfrac{1}{\cos^2\alpha}$ and $1 + \cot^2\alpha = \dfrac{1}{\sin^2\alpha}$, where $\cot\alpha = \cos\alpha/\sin\alpha$.
- **Reduction rule:** Odd multiples of $\frac{\pi}{2}$ change the function ($\sin \leftrightarrow \cos$); even multiples keep it the same.
- **Sign rule:** Assume α is acute, determine the quadrant, and use the sign chart.
- These identities are essential for simplifying expressions and solving equations.

Common Mistakes

- **Forgetting the sign:** The most common error in reduction formulas is getting the sign wrong. Always check which quadrant the shifted angle is in.
- **Confusing function change:** Remember: odd multiples of $\frac{\pi}{2}$ change the function, even multiples don't. Don't mix this up!
- **Ignoring the quadrant when finding values:** If you know $\sin^2\alpha = \frac{9}{25}$, then $\sin\alpha = \pm\frac{3}{5}$. You need quadrant information to determine the correct sign.
- **Forgetting that $\tan$ has period π:** When reducing tangent, remember that $\tan(\alpha + \pi) = \tan\alpha$ (no sign change for full π shift).

Classwork

1. Using $\tan x = \dfrac{\sin x}{\cos x}$, find the exact value of each:
   (a) $\tan \dfrac{2\pi}{3}$ (b) $\tan \dfrac{5\pi}{4}$ (c) $\tan \dfrac{7\pi}{6}$ (d) $\tan \dfrac{11\pi}{6}$
2. Use the Pythagorean identity to find the remaining values:
   (a) $\sin \alpha = \dfrac{3}{5}$ and α is in Q2. Find $\cos \alpha$ and $\tan \alpha$.
   (b) $\tan \alpha = -2$ and $\dfrac{\pi}{2} < \alpha < \pi$. Find $\sin \alpha$ and $\cos \alpha$.
3. Simplify each expression using reduction formulas:
   (a) $\sin\left(\dfrac{\pi}{2} + x\right) + \sin\left(\dfrac{\pi}{2} - x\right)$ (b) $\cos(\pi + x) - \cos(\pi - x)$ (c) $\tan(\pi - x) + \tan(\pi + x)$
4. Find all x in $[0, 2\pi]$ such that:
   (a) $\tan x = 1$
   (b) $\tan x = -\sqrt{3}$
5. M Given $\sin \alpha = p$ where $0 < p < 1$ and $0 < \alpha < \dfrac{\pi}{2}$, express each in terms of p:
   (a) $\cos \alpha$ (b) $\tan \alpha$ (c) $\sin(\pi - \alpha)$ (d) $\cos\left(\dfrac{\pi}{2} + \alpha\right)$

Classwork Solutions

1. (a) Q2: $\tan \frac{2\pi}{3} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$
   (b) Q3: $\tan \frac{5\pi}{4} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$
   (c) Q3: $\tan \frac{7\pi}{6} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$
   (d) Q4: $\tan \frac{11\pi}{6} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$
2. (a) $\cos^2 \alpha = 1 - \frac{9}{25} = \frac{16}{25}$. Q2: $\cos \alpha = -\frac{4}{5}$, $\tan \alpha = \frac{3/5}{-4/5} = -\frac{3}{4}$.
   (b) $1 + 4 = \frac{1}{\cos^2 \alpha}$, so $\cos^2 \alpha = \frac{1}{5}$. Q2: $\cos \alpha = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$.
   $\sin \alpha = \tan \alpha \cdot \cos \alpha = (-2)\left(-\frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$.
3. (a) $\cos x + \cos x = 2\cos x$
   (b) $-\cos x - (-\cos x) = 0$
   (c) $-\tan x + \tan x = 0$
4. (a) $\tan x = 1$: reference angle $\frac{\pi}{4}$, Q1 and Q3. $x = \frac{\pi}{4}, \frac{5\pi}{4}$.
   (b) $\tan x = -\sqrt{3}$: reference angle $\frac{\pi}{3}$, Q2 and Q4. $x = \frac{2\pi}{3}, \frac{5\pi}{3}$.
5. (a) $\cos \alpha = \sqrt{1 - p^2}$
   (b) $\tan \alpha = \dfrac{p}{\sqrt{1 - p^2}}$
   (c) $\sin(\pi - \alpha) = \sin \alpha = p$
   (d) $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha = -p$

Homework

1. Reduce the following expressions to functions of α only:

   (a) $\cos\left(x - \frac{\pi}{2}\right)$ (c) $\tan(2\pi - x)$ (e) $\sin\left(\frac{3\pi}{2} - x\right)$

   (b) $\sin(\pi + x)$ (d) $\cos\left(x + \frac{3\pi}{2}\right)$ (f) $\cot\left(x + \frac{3\pi}{2}\right)$

2. If $\sin x = \frac{5}{13}$ and $0 < x < \frac{\pi}{2}$, find:
   (a) $\cos\left(x + \frac{\pi}{2}\right)$
   (b) $\cot\left(\frac{3\pi}{2} - x\right)$
   (c) $\sin(2\pi - x)$

3. It is known that $\tan\alpha + \cot\alpha = 4$. Without finding α explicitly:
   (a) Find $\sin\alpha\cos\alpha$.
   (b) Find $\tan^2\alpha + \cot^2\alpha$.

4. Simplify each expression:
   (a) $\sin^2 x + \cos^2\left(x + \frac{\pi}{2}\right)$
   (b) $\cos(\pi - x) + \cos(-x)$
   (c) $\sin\left(\frac{\pi}{2} + x\right) - \cos\left(\frac{\pi}{2} - x\right)$
   (d) $\dfrac{\sin(\pi + x) \cdot \tan\left(\dfrac{\pi}{2} - x\right)}{\cos(2\pi - x) \cdot \cot(\pi + x)}$

5. M Show that:

$$\sin x + \sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{2\pi}{3}\right) + \sin(x + \pi) + \sin\left(x + \frac{4\pi}{3}\right) + \sin\left(x + \frac{5\pi}{3}\right) = 0.$$

   (Hint: pair each term with the term that is π away from it.)

6. Find all $x \in [0, 2\pi]$ such that

$$2\sin^2 x = 1 - \cos(\pi + x).$$

7. M Prove the identity:

$$\frac{\sin(\pi + x)}{\cos\left(\frac{\pi}{2} + x\right)} + \frac{\cos(2\pi - x)}{\sin\left(\frac{3\pi}{2} - x\right)} = 0.$$

Quick Check Answers

Tangent in the Trigonometric Circle

1. $\tan\alpha = \sin\alpha/\cos\alpha$; when $\cos\alpha = 0$ the denominator is zero (undefined). Geometrically, the radius is vertical — parallel to the tangent line $x = 1$ — so they never meet.
2. Positive in Q1 ($\sin > 0$, $\cos > 0$) and Q3 ($\sin < 0$, $\cos < 0$). Negative in Q2 and Q4.
3. $\tan(\pi/6) = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$; $\quad \tan(\pi/4) = 1$; $\quad \tan(\pi/3) = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$.

Graph of the Tangent Function

1. $\tan x = 0$ at $x = k\pi$. In $(-2\pi, 2\pi)$: $x = -\pi,\ 0,\ \pi$.
2. $\tan x$ undefined at $x = \frac{\pi}{2} + k\pi$. In $(-2\pi, 2\pi)$: $x = -\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2}$.
3. Period π. Each branch on the graph is an exact horizontal copy of the adjacent one, shifted by π.

The Pythagorean Identity

1. $\cos^2\alpha = 1 - \frac{25}{169} = \frac{144}{169}$, so $\sin\alpha = \frac{12}{13}$ (Q1, positive).
2. $\cos^2\alpha = 1 - \frac{1}{4} = \frac{3}{4}$, so $\cos\alpha = -\frac{\sqrt{3}}{2}$ (Q3, negative).
3. $1 + 4 = \frac{1}{\cos^2\alpha}$, so $\cos\alpha = \frac{1}{\sqrt{5}}$. Then $\sin\alpha = 2 \cdot \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}}$.

Reduction Formulas

1. $\cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) = \sin x$
2. $\sin(\pi + x) = -\sin x$
3. $\tan(2\pi - x) = \tan(-x) = -\tan x$
4. $\cos\left(x + \frac{3\pi}{2}\right) = \sin x$