MATH 7: HANDOUT 23

TRIGONOMETRY IV: TANGENT, IDENTITIES, AND REDUCTION FORMULAS (SUMMARY)

Tangent: Definition and Geometric Interpretation

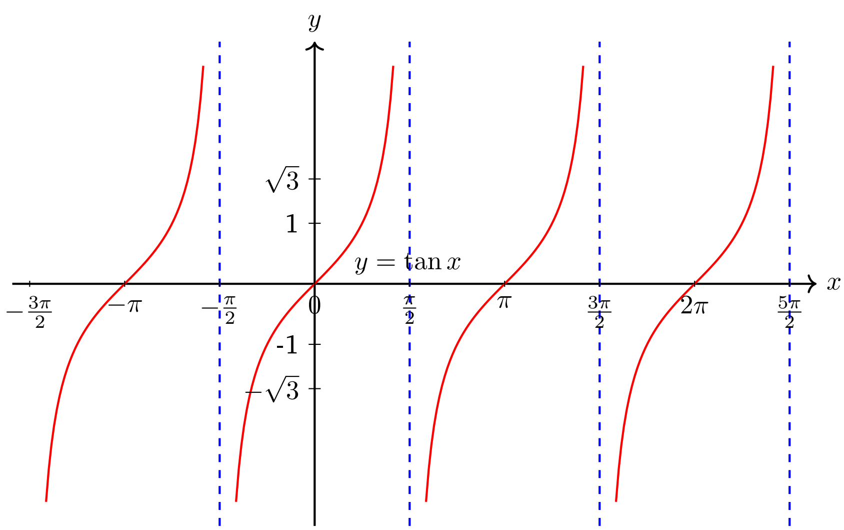
The **tangent** of an angle α is defined by

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \qquad (\cos\alpha \neq 0).$$

Geometrically: draw the radius to the angle α on the unit circle and extend it to the vertical line $x = 1$. The y-coordinate of the intersection is $\tan\alpha$. When $\cos\alpha = 0$ the radius is vertical and never meets the line $x = 1$, so $\tan\alpha$ is undefined.

Key Facts About the Tangent Graph

- **Period:** π (i.e., $\tan(x+\pi) = \tan x$ for all x).
- **Zeros:** $\tan x = 0$ exactly when $x = k\pi$, $k \in \mathbb{Z}$.
- **Vertical asymptotes:** at $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, where $\tan x$ is undefined.
- **Sign by quadrant:** positive in Q1 ($\sin > 0$, $\cos > 0$) and Q3 ($\sin < 0$, $\cos < 0$); negative in Q2 and Q4.
- On each interval $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, the function $\tan x$ is strictly increasing from $-\infty$ to $+\infty$.



The Pythagorean Identity and Its Consequences

$$\boxed{\sin^2\alpha + \cos^2\alpha = 1}$$

Consequence 1: $\cos^2\alpha = 1 - \sin^2\alpha$ **Consequence 2:** $\sin^2\alpha = 1 - \cos^2\alpha$

Given one trig value and the quadrant, the identity determines all the others (up to sign from the quadrant).

Cotangent and Related Identities

The **cotangent** is defined as

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\tan\alpha} \qquad (\sin\alpha \neq 0).$$

Dividing the Pythagorean identity by $\cos^2\alpha$ (resp. $\sin^2\alpha$) gives:

$$\boxed{1+\tan^2\alpha = \frac{1}{\cos^2\alpha}} \qquad \boxed{1+\cot^2\alpha = \frac{1}{\sin^2\alpha}}$$

Reduction Formulas

General Rule (2 steps)

1. **Does the function change?**
   - Shift is an *odd* multiple of $\frac{\pi}{2}$ (e.g. $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$): **function changes** ($\sin \leftrightarrow \cos$, $\tan \leftrightarrow \cot$).
   - Shift is an *even* multiple of $\frac{\pi}{2}$ (e.g. $\pm\pi$, $\pm 2\pi$): **function stays the same**.
2. **What is the sign?** Treat α as a small acute angle. Find which quadrant the shifted angle falls in. The sign is the sign of the (possibly new) function in that quadrant.

Summary Table

Angle	$\sin$	$\cos$	$\tan$
$-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$
$\frac{\pi}{2}-\alpha$	$\cos\alpha$	$\sin\alpha$	$\cot\alpha$
$\frac{\pi}{2}+\alpha$	$\cos\alpha$	$-\sin\alpha$	$-\cot\alpha$
$\pi-\alpha$	$\sin\alpha$	$-\cos\alpha$	$-\tan\alpha$
$\pi+\alpha$	$-\sin\alpha$	$-\cos\alpha$	$\tan\alpha$
$\frac{3\pi}{2}-\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\cot\alpha$
$\frac{3\pi}{2}+\alpha$	$-\cos\alpha$	$\sin\alpha$	$-\cot\alpha$
$2\pi-\alpha$	$-\sin\alpha$	$\cos\alpha$	$-\tan\alpha$

Key Takeaways:

- $\sin^2\alpha + \cos^2\alpha = 1$ lets you find any trig value from one value + quadrant.
- $1+\tan^2\alpha = 1/\cos^2\alpha$ and $1+\cot^2\alpha = 1/\sin^2\alpha$ follow by dividing the Pythagorean identity.
- Odd multiples of $\frac{\pi}{2}$ swap sin/cos (and tan/cot); even multiples keep the function.
- Always determine the quadrant to get the correct sign.
- $\tan$ has period π: $\tan(\alpha+\pi) = \tan\alpha$ (no sign change).

Common Mistakes:

- **Wrong sign in reduction:** Always check the quadrant of the shifted angle.
- **Forgetting function change:** Odd multiple of $\frac{\pi}{2}$ *does* change the function; even multiple does *not*.
- **Ignoring quadrant for ± sign:** $\cos^2\alpha = \frac{16}{25}$ gives $\cos\alpha = \pm\frac{4}{5}$; the quadrant decides which.
- **Confusing period of tan vs sin/cos:** $\tan$ has period π; $\sin$ and $\cos$ have period 2π.

Homework

1. Reduce the following expressions to functions of x only:
   (a) $\cos\left(x - \frac{\pi}{2}\right)$
   (b) $\sin(\pi + x)$
   (c) $\tan(2\pi - x)$
   (d) $\cos\left(x + \frac{3\pi}{2}\right)$
   (e) $\sin\left(\frac{3\pi}{2} - x\right)$
   (f) $\cot\left(x + \frac{3\pi}{2}\right)$
2. If $\sin x = \frac{5}{13}$ and $0 < x < \frac{\pi}{2}$, find:
   (a) $\cos\left(x + \frac{\pi}{2}\right)$
   (b) $\cot\left(\frac{3\pi}{2} - x\right)$
   (c) $\sin(2\pi - x)$
3. It is known that $\tan\alpha + \cot\alpha = 4$. Without finding α explicitly:
   (a) Find $\sin\alpha\cos\alpha$.
   (b) Find $\tan^2\alpha + \cot^2\alpha$.
4. Simplify each expression:
   (a) $\sin^2 x + \cos^2\left(x + \frac{\pi}{2}\right)$
   (b) $\cos(\pi - x) + \cos(-x)$
   (c) $\sin\left(\frac{\pi}{2} + x\right) - \cos\left(\frac{\pi}{2} - x\right)$
   (d) $\dfrac{\sin(\pi + x) \cdot \tan\left(\frac{\pi}{2} - x\right)}{\cos(2\pi - x) \cdot \cot(\pi + x)}$
5. M Show that:
$$\sin x + \sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{2\pi}{3}\right) + \sin(x + \pi) + \sin\left(x + \frac{4\pi}{3}\right) + \sin\left(x + \frac{5\pi}{3}\right) = 0.$$
   (Hint: pair each term with the term that is π away from it.)
6. Find all $x \in [0, 2\pi]$ such that
$$2\sin^2 x = 1 - \cos(\pi + x).$$
7. M Prove the identity:
$$\frac{\sin(\pi + x)}{\cos\left(\frac{\pi}{2} + x\right)} + \frac{\cos(2\pi - x)}{\sin\left(\frac{3\pi}{2} - x\right)} = 0.$$