MATH 7: HANDOUT 22

TRIGONOMETRY III: RADIANS. UNIT CIRCLE. GRAPHS OF TRIGONOMETRIC FUNCTIONS.

Radians

Up to this point, we have been measuring angles in *degrees*, where a full turn is defined to be $360°$. Degrees are convenient and familiar, but not always the most natural choice in mathematics.

Definition. Place an angle α at the center of a circle of radius R. The *radian measure* of α is

$$\text{radians} = \frac{\text{length of the arc cut by } \alpha}{R}.$$

This ratio depends only on the angle, not on the size of the circle.

In particular, if $R = 1$, the *radian measure* of α is

$$\text{radians} = \text{length of the arc cut by } \alpha.$$

Since the circumference of a circle is $2\pi R$, the full angle $360°$ corresponds to

$$360° \longleftrightarrow \frac{2\pi R}{R} = 2\pi \text{ radians.}$$

Thus

$$0° = 0, \qquad 30° = \frac{\pi}{6}, \qquad 45° = \frac{\pi}{4}, \qquad 60° = \frac{\pi}{3}, \qquad 90° = \frac{\pi}{2}.$$

Quick Check

1. Convert to radians:

$$120°, \quad 135°, \quad 210°.$$

2. Convert to degrees:

$$\frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{5\pi}{4}.$$

3. On a unit circle, what is the length of the arc corresponding to:

$$\frac{\pi}{6}, \quad \frac{\pi}{2}, \quad \frac{3\pi}{2}?$$

Measuring Angles: Degrees, Radians, and Other Systems

An angle is simply a measure of rotation, and there is more than one way to describe how “far” you turn. Several systems have been used throughout history and in different sciences.

- **Degrees.** The most familiar system divides a full turn into $360°$. This number is historical: ancient astronomers liked 360 because it has many divisors and is close to the number of days in a year. Degrees are convenient in everyday life, but the number 360 has no special geometric meaning.

- **Radians.** Radians come directly from the geometry of the circle. To measure an angle in radians, we compare it to the length of the arc it cuts out on a *unit* circle (a circle of radius 1). A full turn has arc length 2π, so a complete rotation is 2π radians. Because radians are defined using arc length, they work naturally with trigonometric functions and make many formulas simpler.

- **Gradians (gons).** Used in surveying and some engineering fields, this system divides a full turn into 400 gradians. A right angle is exactly 100 gradians. Gradians were introduced during the French Revolution as part of an attempt to create fully decimal measurement systems.

- **Turns (revolutions).** A very simple system: a full turn is 1 turn. A half turn is $\frac{1}{2}$, a right angle is $\frac{1}{4}$. This system appears in robotics, computer graphics, and higher-level mathematics.

Each system has its uses, but **radians** have a special geometric meaning: they measure rotation using arc length on the unit circle. Because of this, radians behave naturally with sine, cosine, and all other trigonometric ideas we develop in this course. For this reason, mathematicians and scientists almost always use radians when working with trigonometric functions.

Common Trigonometric Values (Radians)

Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\alpha$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

These special values will be our main reference points.

Quick Check

4. Using the table, compute:
$$\sin\left(\frac{\pi}{3}\right), \quad \cos\left(\frac{\pi}{6}\right), \quad \tan\left(\frac{\pi}{4}\right).$$
5. Which of the angles
$$0,\ \frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3},\ \frac{\pi}{2}$$
has the largest value of $\sin\alpha$? Which has the smallest value of $\cos\alpha$?
6. For which angles in the table is $\tan\alpha$ undefined? Why does that happen?

The Trigonometric (Unit) Circle

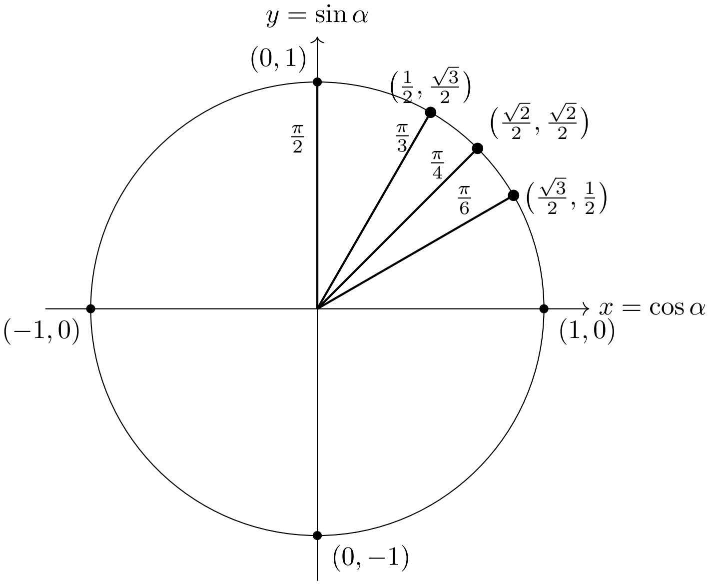
Another powerful way to understand trigonometric functions is through the *unit circle*.

Definition. The unit circle is the circle of radius 1 centered at the origin.

To find $\sin\alpha$ and $\cos\alpha$ using the unit circle:

1. Start at the point $(1, 0)$.
2. Walk around the circle counterclockwise if $\alpha > 0$ and clockwise if $\alpha < 0$.
3. The point you reach has coordinates
$$(\cos\alpha,\ \sin\alpha).$$

Thus, cosine is the x-coordinate and sine is the y-coordinate of a point on the unit circle. This also explains why both sine and cosine are always between -1 and 1.



Quick Check

7. A point on the unit circle has coordinates

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Which angle (in radians between 0 and 2π) does this correspond to?

8. A point on the unit circle has coordinates

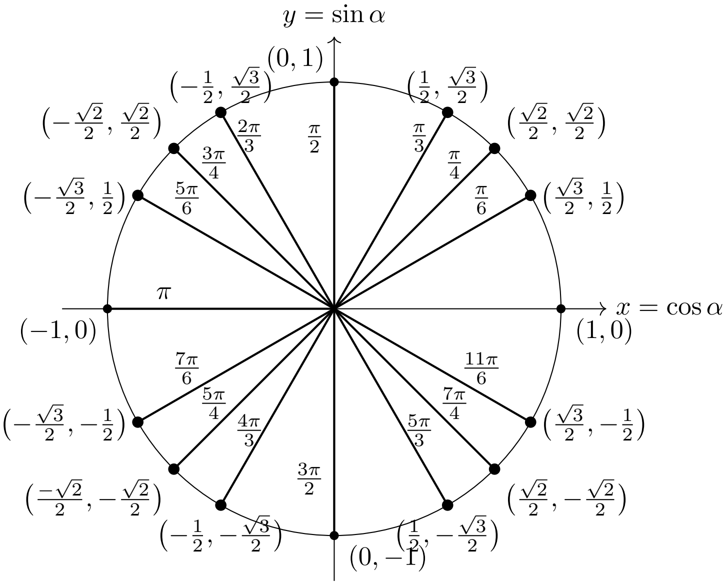
$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

Which angle (in radians between 0 and 2π) does this correspond to?

9. In which quadrants is $\sin x$ positive? In which quadrants is $\cos x$ positive?

Sine and Cosine for All Angles

The same definition extends to *any* real number x: for negative x, walk clockwise; for $x > 2\pi$, continue past one full turn. The diagram below shows all four quadrants.



Quick Check

10. Use the unit circle to find (no calculator):

$$\sin\left(\frac{7\pi}{6}\right), \quad \cos\left(\frac{5\pi}{4}\right), \quad \sin\left(-\frac{\pi}{3}\right), \quad \cos\left(\frac{11\pi}{6}\right).$$

11. Find all solutions x in the interval $[0, 2\pi]$:
    (a) $\sin x = \frac{\sqrt{3}}{2}$,
    (b) $\sin x = -\frac{1}{2}$,
    (c) $\cos x = -\frac{\sqrt{2}}{2}$,
    (d) $\cos x = 0$.

12. Without graphing, determine the sign (positive or negative) of each expression:

$$\sin\left(\frac{4\pi}{3}\right), \qquad \cos\left(\frac{5\pi}{6}\right), \qquad \sin\left(\frac{9\pi}{4}\right), \qquad \cos\left(-\frac{\pi}{2}\right).$$

Basic Properties from the Unit Circle

By watching how the point $(\cos x, \sin x)$ moves around the unit circle as x increases, several important facts become visually obvious:

- **Periodicity.** After a full turn 2π, you are back at the same point on the circle, so

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x + 2\pi) = \cos x.$$

- **Half-turn symmetry.** A rotation by π (half a circle) sends each point to the opposite side of the circle, which flips both coordinates:

$$\sin(x + \pi) = -\sin x, \qquad \cos(x + \pi) = -\cos x.$$

- **Zeros, maxima, and minima** ($k \in \mathbb{Z}$):

	$\sin x$	$\cos x$
Zeros $(=0)$ at	$x = k\pi$	$x = \frac{\pi}{2} + k\pi$
Maximum $(=1)$ at	$x = \frac{\pi}{2} + 2k\pi$	$x = 2k\pi$
Minimum $(=-1)$ at	$x = \frac{3\pi}{2} + 2k\pi$	$x = \pi + 2k\pi$

Symmetry of the Trigonometric Functions

The unit circle makes the symmetry of $\sin x$ and $\cos x$ very easy to see.

- **Sine:**
  $$\sin(-x) = -\sin x.$$
  This is because the y-coordinate is positive above the x-axis and negative below it.
- **Cosine:**
  $$\cos(-x) = \cos x.$$
  Reflection across the x-axis does not change the x-coordinate.

Graphically, this means:

sine is symmetric with respect to the origin,   cosine is symmetric with respect to the y-axis.

These facts become especially useful when solving equations and sketching graphs.

Quick Check

13. Find all x in $[0, 2\pi]$ such that:
    (a) $\sin x = 0$,
    (b) $\cos x = 0$,
    (c) $\cos x = 1$.
14. True or false (justify using the unit circle):
    (a) $\sin(x + 2\pi) = \sin x$ for every real x
    (b) $\cos(x + 2\pi) = -\cos x$ for every real x
    (c) $\cos(x + \pi) = \cos x$ for every real x
    (d) $\sin(x + \pi) = -\sin x$ for every real x
    (e) $\sin x = 0$ exactly when $x = k\pi$
    (f) $\cos x = 1$ exactly when $x = 2k\pi$

Graphs of the Sine and Cosine Functions

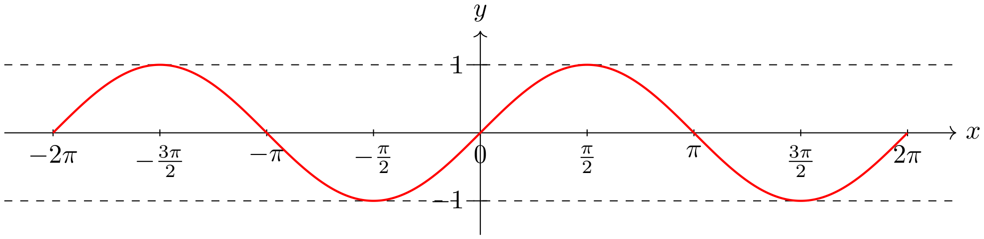
Sine

By looking at the values of $\sin x$ as we go around the unit (trigonometric) circle, we can observe several important facts:

- $\sin 0 = 0$ and $\sin \pi = 0$.
- On the interval $0 \le x \le \frac{\pi}{2}$, the function $\sin x$ increases from 0 to 1.
- At $x = \frac{\pi}{2}$, $\sin x$ reaches its maximum value 1.

- At $x = \frac{3\pi}{2}$, $\sin x$ reaches its minimum value -1.
- For all real x, $\sin(x + 2\pi) = \sin x$, so the graph repeats every 2π.

We can see all of these features clearly in the graph of $y = \sin x$:

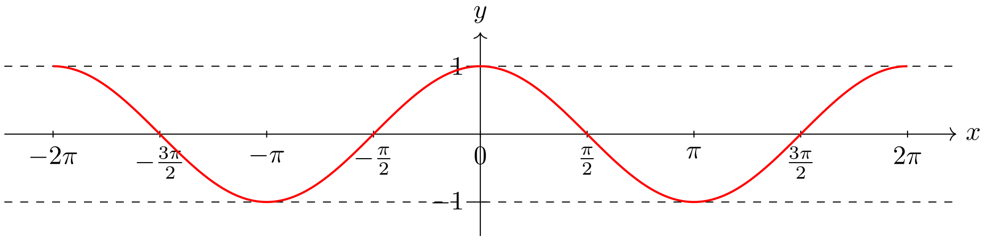


Cosine

Now, by looking at the values of $\cos x$ as we go around the unit circle, we can observe several important facts:

- $\cos 0 = 1$ and $\cos \pi = -1$.
- $\cos x = 0$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
- On the interval $0 \leq x \leq \pi$, the function $\cos x$ decreases from 1 to -1.
- On the interval $\pi \leq x \leq 2\pi$, the function $\cos x$ increases from -1 back to 1.
- For all real x, $\cos(x + 2\pi) = \cos x$, so the graph repeats every 2π.

We can see all of these features clearly in the graph of $y = \cos x$:



Quick Check

15. Using the graph of $y = \sin x$, find all x in $[0, 2\pi]$ where:
    (a) $\sin x = 0$,
    (b) $\sin x = 1$,
    (c) $\sin x = -1$.
16. Using the graph of $y = \cos x$, find all x in $[0, 2\pi]$ where:
    (a) $\cos x = 0$,
    (b) $\cos x = 1$,
    (c) $\cos x = -1$.

Where You've Seen Trigonometry Before

Trigonometry appears everywhere, long before students ever hear the word.

- **Shadows and Sunlight.** Ancient mathematicians measured the height of tall objects (trees, pyramids) by comparing their shadows and using angle ratios.
- **Phone Tilt Sensors.** Your phone's accelerometer measures the pull of gravity in different directions. The operating system uses $\sin$ and $\cos$ to turn that information into screen orientation.
- **GPS and Satellites.** Every GPS receiver uses the angles from at least four satellites to locate your position on Earth, a giant real-world triangle.
- **Waves.** Sound, music, light, radio, and even Wi-Fi signals behave like sine curves.

Trigonometry isn't just about triangles—it is the language of motion, light, and measurement.

Summary: The Six Trigonometric Functions

On the unit circle: If a point on the unit circle has coordinates $(x, y) = (\cos\theta, \sin\theta)$, then

$$\sin\theta = y, \qquad \cos\theta = x, \qquad \tan\theta = \frac{y}{x} \quad (\text{when } x \neq 0).$$

Key identities:

$$\sin^2\theta + \cos^2\theta = 1.$$

Periodicity:

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta, \qquad \tan(\theta + \pi) = \tan\theta.$$

Symmetry:

$$\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta, \qquad \tan(-\theta) = -\tan\theta.$$

Key Takeaways

- **Radians:** A full circle is 2π radians. To convert, use $180° = \pi$ radians.
- **Unit circle:** For any angle α, the point on the unit circle is $(\cos\alpha, \sin\alpha)$.
- **All quadrants:** Use the unit circle to extend sine and cosine to all angles, not just acute angles.
- **Periodicity:** $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$.
- **Symmetry:** $\sin(-x) = -\sin x$ (odd function), $\cos(-x) = \cos x$ (even function).
- **Range:** $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$.

Common Mistakes

- **Mixing up degrees and radians:** When your calculator is in the wrong mode, you get wrong answers. Always check! In pure mathematics, radians are standard.
- **Forgetting which quadrant you're in:** In quadrant II, $\sin > 0$ but $\cos < 0$. In quadrant III, both are negative. In quadrant IV, $\cos > 0$ but $\sin < 0$. Draw the unit circle to check signs.

Classwork

1. Convert to radians: $150°$, $240°$, $330°$. Convert to degrees: $\frac{5\pi}{6}$, $\frac{7\pi}{4}$, $\frac{11\pi}{6}$.
2. Using the unit circle, find the exact values of $\sin x$ and $\cos x$ for:
   (a) $x = \frac{3\pi}{4}$
   (b) $x = \frac{5\pi}{3}$
   (c) $x = \frac{7\pi}{6}$
3. Simplify using periodicity and symmetry:
   (a) $\sin(x + 2\pi)$
   (b) $\cos(-x)$
   (c) $\sin(x + \pi)$
   (d) $\tan(x + \pi)$
4. Find all x in $[0, 2\pi]$ such that:
   (a) $\sin x = \frac{1}{2}$
   (b) $\cos x = -\frac{\sqrt{3}}{2}$
   (c) $\tan x = -1$
5. On which intervals in $[0, 2\pi]$ is $\sin x > 0$? On which is $\sin x < 0$? Answer the same questions for $\cos x$.

Classwork Solutions

1. $150° = \frac{5\pi}{6}$, $\quad 240° = \frac{4\pi}{3}$, $\quad 330° = \frac{11\pi}{6}$.
   $\frac{5\pi}{6} = 150°$, $\quad \frac{7\pi}{4} = 315°$, $\quad \frac{11\pi}{6} = 330°$.
2. (a) $\frac{3\pi}{4}$ (Q2): $\sin = \frac{\sqrt{2}}{2}$, $\cos = -\frac{\sqrt{2}}{2}$
   (b) $\frac{5\pi}{3}$ (Q4): $\sin = -\frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$
   (c) $\frac{7\pi}{6}$ (Q3): $\sin = -\frac{1}{2}$, $\cos = -\frac{\sqrt{3}}{2}$
3. (a) $\sin x$
   (b) $\cos x$
   (c) $-\sin x$
   (d) $\tan x$
4. (a) $x = \frac{\pi}{6}, \frac{5\pi}{6}$
   (b) $x = \frac{5\pi}{6}, \frac{7\pi}{6}$
   (c) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

5. $\sin x > 0$ on $(0, \pi)$; $\sin x < 0$ on $(\pi, 2\pi)$.
   $\cos x > 0$ on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$; $\cos x < 0$ on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Homework

1. Complete the table below. For each angle, convert between degrees and radians where needed, reduce to $[0, 2\pi)$ if necessary, then find the exact values. The first row is done as an example. Leave answers in exact form (no decimals).

Degrees	Radians	$\sin x$	$\cos x$	$\tan x$
$180°$	π	0	-1	0
$45°$				
$120°$				
$330°$				
	$\frac{7\pi}{6}$			
	$\frac{5\pi}{3}$			
	$-\frac{\pi}{4}$			
	11π			
	$\frac{25\pi}{3}$			
	$-\frac{19\pi}{6}$			
	$\frac{9\pi}{4}$			
		$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
		$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
		$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1

2. Using the trigonometric circle, show that

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

for any angle x. Then use this fact and the graph of the sine function to construct (draw) the graph of the cosine function.

3. Find all real numbers x such that

$$(\sin x)^2 = \frac{3}{4}.$$

4. Using the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$:
    - (a) Find all solutions of $\sin x = \frac{\sqrt{3}}{2}$ in the interval $[0, 2\pi]$.
    - (b) Find all solutions of $\cos x = -\frac{1}{2}$ in the interval $[-2\pi, 2\pi]$.
    - (c) Find all solutions of $\tan x = \sqrt{3}$ in the interval $(-\pi, \pi)$.

5. M Solve the following equation without a calculator:

$$\sin(3x) = \frac{\sqrt{3}}{2}.$$

Find all solutions in the interval $[0, 2\pi]$. (Hint: first solve $3x = \frac{\pi}{3} + 2k\pi$ and $3x = \frac{2\pi}{3} + 2k\pi$.)

6. A point on the unit circle has y-coordinate

$$\sin x = -\frac{\sqrt{2}}{2}.$$

- (a) Find all such angles x in $[-2\pi, 2\pi]$.
- (b) How many of these angles lie in Quadrant III?

7. M Two different angles A and B satisfy

$$\sin A = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos B = -\frac{1}{2}.$$

- (a) Find all possible values of A in $[0, 2\pi]$.
- (b) Find all possible values of B in $[0, 2\pi]$.
- (c) List *all* possible values of $A - B$ in $[0, 2\pi]$.

(Hint: use the unit circle and quadrant signs.)