MATH 7: HANDOUT 22

TRIGONOMETRY III: RADIANS, UNIT CIRCLE, AND GRAPHS (SUMMARY)

Radians

Definition 1 (Radian Measure). Place angle α at the center of a circle of radius R. The radian measure is $\alpha\ (\mathrm{rad}) = \dfrac{\text{arc length}}{R}$. On the unit circle ($R = 1$), the radian measure equals the arc length cut by α.

Since a full circle has arc length $2\pi R$, a full turn equals 2π radians:

$$180^\circ = \pi \text{ rad.} \qquad \text{To convert: multiply by } \tfrac{\pi}{180} \text{ (deg}\to\text{rad) or } \tfrac{180}{\pi} \text{ (rad}\to\text{deg).}$$

Example: $120^\circ = 120 \cdot \dfrac{\pi}{180} = \dfrac{2\pi}{3}$ **Example:** $\dfrac{5\pi}{4} = \dfrac{5\pi}{4} \cdot \dfrac{180}{\pi} = 225^\circ$

Key values: $30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, $90^\circ = \frac{\pi}{2}$, $180^\circ = \pi$.

Common Trigonometric Values

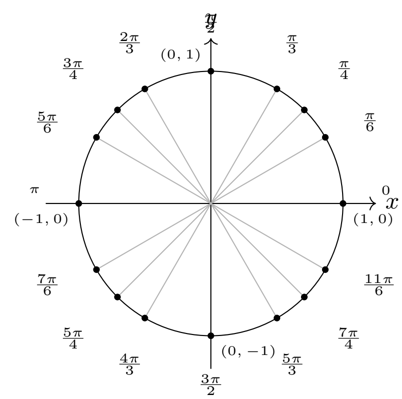
Angle	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\alpha$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

For other angles, use the sign table and the reference angle in Q1:

The Unit Circle

Walk counterclockwise from $(1,0)$ a distance x; the point reached is $(\cos x, \sin x)$.

	Q1	**Q2**	**Q3**	**Q4**
sin	$+$	$+$	$-$	$-$
cos	$+$	$-$	$-$	$+$
tan	$+$	$-$	$+$	$-$



Key Properties

Periodicity and half-turn:

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x + 2\pi) = \cos x, \qquad \sin(x + \pi) = -\sin x, \qquad \cos(x + \pi) = -\cos x.$$

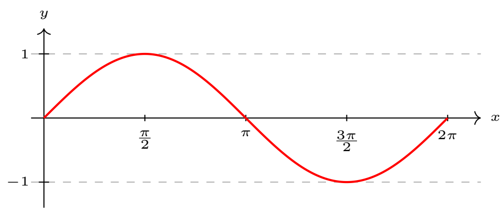
Symmetry (odd/even):

$$\sin(-x) = -\sin x \quad \text{(odd)}, \qquad \cos(-x) = \cos x \quad \text{(even)}.$$

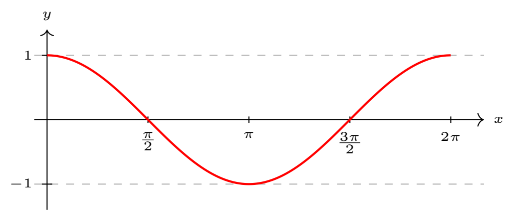
Zeros, maxima, and minima ($k \in \mathbb{Z}$):

	$\sin x$	$\cos x$
Zeros $(= 0)$ at	$x = k\pi$	$x = \frac{\pi}{2} + k\pi$
Maximum $(= 1)$ at	$x = \frac{\pi}{2} + 2k\pi$	$x = 2k\pi$
Minimum $(= -1)$ at	$x = \frac{3\pi}{2} + 2k\pi$	$x = \pi + 2k\pi$

$y = \sin x$



$y = \cos x$



Common Mistakes

- **Mixing degrees and radians.** Check your calculator mode. In pure mathematics, radians are standard.
- **Forgetting quadrant signs.** Always identify the quadrant and apply the sign table above.

Homework

1. Complete the table below. For each angle, convert between degrees and radians where needed, reduce to $[0, 2\pi)$ if necessary, then find the exact values. The first row is done as an example. Leave answers in exact form (no decimals).

Degrees	Radians	$\sin x$	$\cos x$	$\tan x$
$180°$	π	0	-1	0
$45°$				
$120°$				
$330°$				
	$\frac{7\pi}{6}$			
	$\frac{5\pi}{3}$			
	$-\frac{\pi}{4}$			
	11π			
	$\frac{25\pi}{3}$			
	$-\frac{19\pi}{6}$			
	$\frac{9\pi}{4}$			
		$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
		$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
		$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1

2. Using the trigonometric circle, show that

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

for any angle x. Then use this fact and the graph of the sine function to construct (draw) the graph of the cosine function.

3. Find all real numbers x such that

$$(\sin x)^2 = \frac{3}{4}.$$

4. Using the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$:

   (a) Find all solutions of $\sin x = \frac{\sqrt{3}}{2}$ in the interval $[0, 2\pi]$.

   (b) Find all solutions of $\cos x = -\frac{1}{2}$ in the interval $[-2\pi, 2\pi]$.

   (c) Find all solutions of $\tan x = \sqrt{3}$ in the interval $(-\pi, \pi)$.

5. M Solve the following equation without a calculator:

$$\sin(3x) = \frac{\sqrt{3}}{2}.$$

   Find all solutions in the interval $[0, 2\pi]$. (Hint: first solve $3x = \frac{\pi}{3} + 2k\pi$ and $3x = \frac{2\pi}{3} + 2k\pi$.)

6. A point on the unit circle has y-coordinate

$$\sin x = -\frac{\sqrt{2}}{2}.$$

   (a) Find all such angles x in $[-2\pi, 2\pi]$.

   (b) How many of these angles lie in Quadrant III?

7. M Two different angles A and B satisfy

$$\sin A = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos B = -\frac{1}{2}.$$

   (a) Find all possible values of A in $[0, 2\pi]$.

   (b) Find all possible values of B in $[0, 2\pi]$.

   (c) List *all* possible values of $A - B$ in $[0, 2\pi]$.

   (Hint: use the unit circle and quadrant signs.)