MATH 7: HANDOUT 21

Trigonometry II: Area, Law of Sines, and Law of Cosines

Area of a Triangle Using Sine

You already know the familiar formula

$$\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}.$$

Sometimes the height is not given explicitly. In that case, it is convenient to express the height using a sine.

Consider a triangle with sides a and b and included angle γ:

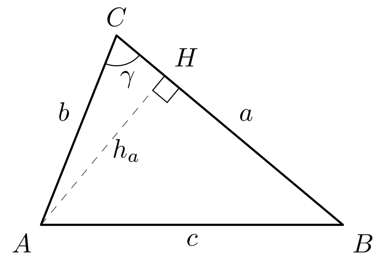
$$\text{Area} = \frac{1}{2} \cdot (\text{base } a) \cdot (\text{height } h_a).$$

From the right triangle $\triangle AHC$, the height is

$$h_a = b \sin \gamma.$$

So

$$\text{Area} = \frac{1}{2}\, a \cdot b \sin \gamma = \frac{1}{2} ab \sin \gamma.$$



By choosing different sides as the base, we get the general formula:

Theorem

Area of a Triangle. For a triangle with sides a, b, c and opposite angles α, β, γ:

$$\text{Area} = \frac{1}{2}\, bc\ \sin \alpha = \frac{1}{2}\, ca\ \sin \beta = \frac{1}{2}\, ab\ \sin \gamma.$$

Example 1. Find the area of a triangle with sides $b = 7$, $c = 10$, and included angle $\alpha = 40°$.

$$\text{Area} = \frac{1}{2}\, bc\ \sin \alpha = \frac{1}{2} \cdot 7 \cdot 10 \cdot \sin 40° \approx 22.5.$$

Quick Check

1. A triangle has sides $b = 8$ and $c = 11$ with included angle $A = 65°$. Find its area.
2. Sides are $a = 12$, $c = 15$, angle $B = 75°$. Find the area.

Fundamental Trigonometric Identity

Theorem

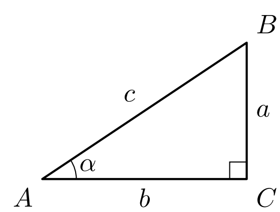
Pythagorean Identity. For any angle α:

$$\sin^2\alpha + \cos^2\alpha = 1.$$

We can prove this using only the definition of sine and cosine in a right triangle and the Pythagorean theorem.

Proof. Consider a right triangle with acute angle α. Let:

- a be the side *opposite* angle α,
- b be the side *adjacent* to angle α,
- c be the *hypotenuse*.



By definition,

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}, \qquad \cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.$$

Now square both:

$$\sin^2\alpha = \left(\frac{a}{c}\right)^2 = \frac{a^2}{c^2}, \qquad \cos^2\alpha = \left(\frac{b}{c}\right)^2 = \frac{b^2}{c^2}.$$

Add them:

$$\sin^2\alpha + \cos^2\alpha = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2}.$$

By the Pythagorean theorem, $a^2 + b^2 = c^2$. Therefore,

$$\sin^2\alpha + \cos^2\alpha = \frac{c^2}{c^2} = 1.$$

□

Consequences: Sine and Cosine Determine Each Other

The identity

$$\sin^2\alpha + \cos^2\alpha = 1$$

has a very important consequence:

$$\boxed{\sin^2\alpha = 1 - \cos^2\alpha \quad \text{and} \quad \cos^2\alpha = 1 - \sin^2\alpha.}$$

This means:

If you know $\sin\alpha$, then you automatically know $|\cos\alpha|$,

and if you know $\cos\alpha$, you automatically know $|\sin\alpha|$.

In other words, **sine and cosine determine each other, except for the sign**. (You learn the sign from the geometry or from which quadrant the angle is in — later.)

Example 2. Suppose you know that

$$\cos\alpha = \frac{3}{5}$$

and that α is an acute angle. Find $\sin\alpha$.

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}.$$

Since α is acute, $\sin\alpha > 0$, so

$$\sin\alpha = \frac{4}{5}.$$

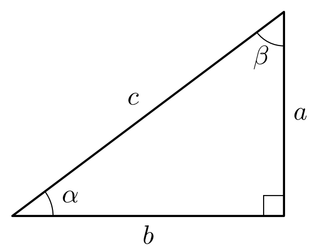
If we did *not* know the angle was acute, then the identity alone gives only

$$\sin\alpha = \pm\frac{4}{5},$$

illustrating that $\cos\alpha$ determines $|\sin\alpha|$ but not the sign by itself.

Complementary Angles

Consider a right triangle with acute angles α and β. Since the angles of a triangle sum to 180 and one angle is 90, we have $\alpha + \beta = 90$, so $\beta = 90 - \alpha$.



From β's perspective:

- The side *opposite* β is b (which was *adjacent* to α)
- The side *adjacent* to β is a (which was *opposite* α)

So:

$$\sin\beta = \frac{b}{c} = \cos\alpha, \qquad \cos\beta = \frac{a}{c} = \sin\alpha$$

Since $\beta = 90 - \alpha$:

Theorem

Complementary Angle Relationships.

$$\sin(90 - \alpha) = \cos\alpha, \qquad \cos(90 - \alpha) = \sin\alpha$$

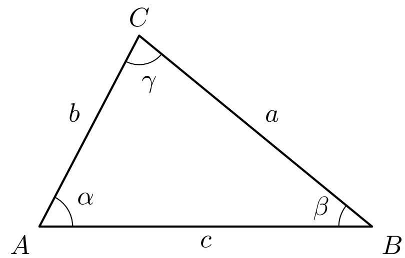
This is why cosine is called “co-sine”—it’s the sine of the **co**mplementary angle!

Quick Check

3. Simplify: $5\sin^2 x + 5\cos^2 x$.
4. Express in terms of $\sin x$ only: $7\sin^2 x + 4\cos^2 x$.
5. Express $\sin^2 x$ in terms of $\cos x$.
6. If $\cos x = \frac{3}{5}$ and x is acute, find $\sin x$.
7. Verify: $\sin 30 = \cos 60$ and $\sin 60 = \cos 30$.
8. Simplify: $\sin 25 \cdot \cos 65 + \cos 25 \cdot \sin 65$. *(Hint: $25 + 65 = 90$.)*

The Law of Sines

Consider triangle ABC with sides a, b, c opposite angles α, β, γ respectively.



Theorem

Law of Sines.

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Proof. Using the area formula from before:

$$\text{Area} = \frac{1}{2}\,bc\ \sin\alpha = \frac{1}{2}\,ca\ \sin\beta = \frac{1}{2}\,ab\ \sin\gamma.$$

Multiplying each by 2:

$$bc\sin\alpha = ca\sin\beta = ab\sin\gamma.$$

Now divide everything by abc:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c},$$

which is equivalent to

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

□

Example 3. In triangle ABC, $A = 30°$, $B = 45°$, and $a = 12$. Find side b.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad\Rightarrow\quad \frac{12}{\sin 30°} = \frac{b}{\sin 45°}.$$

Since $\sin 30° = \frac{1}{2}$ and $\sin 45° = \frac{\sqrt{2}}{2}$,

$$\frac{12}{1/2} = \frac{b}{\sqrt{2}/2} \quad\Rightarrow\quad b = 24\cdot\frac{\sqrt{2}}{2} = 12\sqrt{2}.$$

Example 4. In triangle ABC, $a = 14$, $b = 10$, and $A = 50°$. Find angle B.

Using the Law of Sines, we can get $\sin B$:

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \Rightarrow \quad \sin B = \frac{b\,\sin A}{a} = \frac{10 \sin 50°}{14}.$$

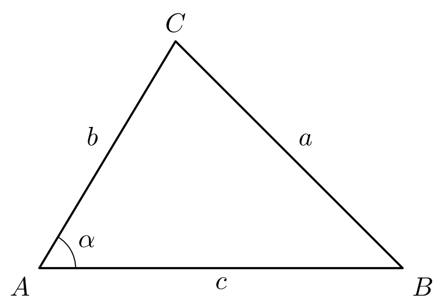
Quick Check

9. In $\triangle ABC$, $\alpha = 40°$, $\beta = 70°$, $a = 9$. Find b.
10. In $\triangle ABC$, $a = 14$, $b = 10$, $\alpha = 50°$. Find angle β.

The Law of Cosines

The Law of Sines is useful when we know two angles and a side, or two sides and an angle opposite one of them. But what if we know two sides and the *included* angle? Or all three sides but no angles?

For these situations, we need the **Law of Cosines**.



Theorem

Law of Cosines. In any triangle with sides a, b, c and angle α opposite side a:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Similarly: $b^2 = a^2 + c^2 - 2ac\cos\beta$ and $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

Proof. Place the triangle in a coordinate system with vertex A at the origin and B on the positive x-axis. Then:

- $A = (0, 0)$
- $B = (c, 0)$
- $C = (b\cos\alpha, b\sin\alpha)$

Using the distance formula for side $a = BC$:

$$\begin{aligned} a^2 &= (b\cos\alpha - c)^2 + (b\sin\alpha - 0)^2 \\ &= b^2\cos^2\alpha - 2bc\cos\alpha + c^2 + b^2\sin^2\alpha \\ &= b^2(\cos^2\alpha + \sin^2\alpha) + c^2 - 2bc\cos\alpha \\ &= b^2 + c^2 - 2bc\cos\alpha. \end{aligned}$$

□

Note: When $\alpha = 90^\circ$, we have $\cos\alpha = 0$, so the Law of Cosines becomes $a^2 = b^2 + c^2$, which is exactly the Pythagorean theorem. The Law of Cosines is a generalization of the Pythagorean theorem to non-right triangles.

Geometric insight: The term $-2bc\cos\alpha$ is the "correction" to the Pythagorean theorem:

- When $\alpha < 90^\circ$ (acute), $\cos\alpha > 0$, so we *subtract* $2bc\cos\alpha$. This makes $a^2 < b^2 + c^2$: the side opposite an acute angle is shorter than in a right triangle.
- When $\alpha > 90^\circ$ (obtuse), $\cos\alpha < 0$, so we effectively *add* $2bc|\cos\alpha|$. This makes $a^2 > b^2 + c^2$: the side opposite an obtuse angle is longer than in a right triangle.

Note on obtuse angles: We have been working with acute angles so far. What does $\cos\alpha$ mean when $\alpha > 90^\circ$? The definition using right triangles doesn't directly apply! In a future handout, we will extend sine and cosine to *all* angles using the unit circle. For now, you can use a calculator to find values like $\cos 120^\circ = -0.5$, and the Law of Cosines will work correctly. (In fact, the coordinate proof above works for any angle: when $\alpha > 90^\circ$, point C has a negative x-coordinate, and the same algebra still produces the correct result.)

Example 5. In triangle ABC, $b = 7$, $c = 10$, and $\alpha = 60^\circ$. Find side a.

Using the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = 49 + 100 - 2(7)(10)\cos 60^\circ = 149 - 140 \cdot \frac{1}{2} = 149 - 70 = 79.$$

Therefore $a = \sqrt{79} \approx 8.89$.

Example 6. A triangle has sides $a = 5$, $b = 7$, $c = 8$. Find angle C.

Rearranging the Law of Cosines to solve for $\cos C$:

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Substituting:

$$\cos C = \frac{25 + 49 - 64}{2 \cdot 5 \cdot 7} = \frac{10}{70} = \frac{1}{7}.$$

Using a calculator to find the angle whose cosine is $\frac{1}{7}$, we get $C \approx 81.8^\circ$.

Example 7 (Solving a Triangle Completely)**.** In triangle ABC, we know $a = 8$, $b = 6$, and $\gamma = 50^\circ$. Find all remaining sides and angles.

Step 1: Find side c using the Law of Cosines.

$$c^2 = a^2 + b^2 - 2ab\cos\gamma = 64 + 36 - 2(8)(6)\cos 50^\circ = 100 - 96(0.643) \approx 38.3.$$

So $c \approx 6.19$.

Step 2: Find angle α using the Law of Sines.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \quad \Rightarrow \quad \sin\alpha = \frac{a\sin\gamma}{c} = \frac{8 \cdot \sin 50^\circ}{6.19} \approx 0.990.$$

Using a calculator to find the angle, $\alpha \approx 81.9^\circ$.

Step 3: Find angle β using the angle sum.

$$\beta = 180^\circ - \alpha - \gamma \approx 180^\circ - 81.9^\circ - 50^\circ = 48.1^\circ.$$

Summary: $a = 8$, $b = 6$, $c \approx 6.19$, $\alpha \approx 81.9^\circ$, $\beta \approx 48.1^\circ$, $\gamma = 50^\circ$.

Quick Check

11. In triangle ABC, $a = 6$, $b = 8$, $\gamma = 45^\circ$. Find side c.
12. A triangle has sides 4, 5, 6. Find the largest angle.

Key Takeaways

- **Area formula:** The area of a triangle with two sides a, b and included angle γ is

$$\text{Area} = \frac{1}{2}ab\sin\gamma.$$

- **Pythagorean identity:** For any angle α,

$$\sin^2\alpha + \cos^2\alpha = 1.$$

  This means if you know $\sin\alpha$, you can find $|\cos\alpha|$, and vice versa.

- **Law of Sines:** In any triangle with sides a, b, c opposite angles α, β, γ:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

  Use the Law of Sines when you know two angles and one side, or two sides and an angle opposite one of them.

- **Law of Cosines:** $a^2 = b^2 + c^2 - 2bc\cos\alpha$ (and similar formulas for b^2, c^2). Use the Law of Cosines when you know two sides and the included angle, or all three sides.

- The Law of Cosines generalizes the Pythagorean theorem to non-right triangles.

Common Mistakes

- **Using the wrong angle in the area formula:** The angle must be the *included angle* between the two sides you're using. If you have sides a and b, you need angle γ (the angle between them), not angle α or β.

- **Forgetting the absolute value:** The identity $\sin^2\alpha + \cos^2\alpha = 1$ tells you $|\sin\alpha|$ from $\cos\alpha$, not the sign. You need additional information (such as the quadrant) to determine whether $\sin\alpha$ is positive or negative.

- **Law of Sines ambiguity:** When using the Law of Sines to find an angle, you get $\sin\beta = \ldots$, which could give two possible angles (one acute, one obtuse). Check which makes sense for your triangle.

- **Choosing the wrong law:** Use Law of Sines for AAS, ASA, or SSA (ambiguous case). Use Law of Cosines for SAS or SSS.

- **Sign error in Law of Cosines:** Remember it's $-2bc\cos\alpha$, not $+2bc\cos\alpha$. The formula reduces to the Pythagorean theorem when the angle is $90°$.

Classwork

1. Find the area of triangle ABC if $a = 9$, $b = 12$, and $\gamma = 50°$.
2. If $\sin\alpha = \dfrac{5}{13}$ and α is acute, find $\cos\alpha$.
3. In triangle ABC, $\alpha = 35°$, $\beta = 80°$, and $a = 10$. Find sides b and c.
4. In triangle ABC, $a = 5$, $b = 8$, and $\gamma = 60°$. Find side c.
5. A triangle has sides 6, 8, and 11. Find all three angles.
6. The area of a right triangle is 36 m^2. The legs are in the ratio $2 : 9$. Find the length of the hypotenuse.
7. In parallelogram $ABCD$, we are given $AB = 10$, $AD = 4$, and $\angle BAD = 50°$. Find the length of diagonal BD.
8. A regular heptagon (7 sides) is inscribed in a circle of radius 1.
   (a) Find the perimeter of the heptagon.
   (b) Find the area of the heptagon.
9. Two ships leave a port at the same time. One sails 15 km on a bearing of $N30°E$, and the other sails 20 km on a bearing of $N80°E$. How far apart are the ships?

Classwork Solutions

1. Area $= \frac{1}{2}(9)(12)\sin 50° = 54\sin 50° \approx 41.4$
2. $\cos^2\alpha = 1 - \sin^2\alpha = 1 - \frac{25}{169} = \frac{144}{169}$. Since α is acute, $\cos\alpha = \frac{12}{13}$.
3. $\gamma = 180° - 35° - 80° = 65°$. By the Law of Sines:

$$\frac{b}{\sin 80°} = \frac{10}{\sin 35°} \quad\Rightarrow\quad b = \frac{10\sin 80°}{\sin 35°} \approx 17.2.$$

$$\frac{c}{\sin 65°} = \frac{10}{\sin 35°} \quad\Rightarrow\quad c = \frac{10\sin 65°}{\sin 35°} \approx 15.8.$$

4. $c^2 = a^2 + b^2 - 2ab\cos\gamma = 25 + 64 - 80\cos 60° = 89 - 40 = 49$. So $c = 7$.
5. The largest angle is opposite the longest side ($c = 11$):

$$\cos\gamma = \frac{36 + 64 - 121}{2(6)(8)} = \frac{-21}{96} = -\frac{7}{32}, \quad \gamma \approx 102.6°.$$

   For the next angle (opposite $b = 8$):

$$\cos\beta = \frac{36 + 121 - 64}{2(6)(11)} = \frac{93}{132} = \frac{31}{44}, \quad \beta \approx 45.2°.$$

   Then $\alpha = 180° - 102.6° - 45.2° \approx 32.2°$.
6. Let the legs be $2x$ and $9x$. Then:

$$\text{Area} = \frac{1}{2}(2x)(9x) = 9x^2 = 36 \quad\Rightarrow\quad x^2 = 4 \quad\Rightarrow\quad x = 2.$$

   The legs are 4 m and 18 m. The hypotenuse is $\sqrt{16 + 324} = \sqrt{340} = 2\sqrt{85} \approx 18.4$ m.

7. In triangle ABD, we have $AB = 10$, $AD = 4$, and $\angle BAD = 50°$.
   By the Law of Cosines:

$$BD^2 = 100 + 16 - 2(10)(4)\cos 50° = 116 - 80(0.643) \approx 64.6.$$

   So $BD \approx 8.0$.

8. (a) Central angle $= \dfrac{360°}{7}$. Each side $= 2\sin\dfrac{180°}{7} \approx 0.868$.
   Perimeter $= 14\sin\dfrac{180°}{7} \approx 6.07$.
   (b) Area $= \dfrac{7}{2} \cdot 1^2 \cdot \sin\dfrac{360°}{7} \approx 2.74$.

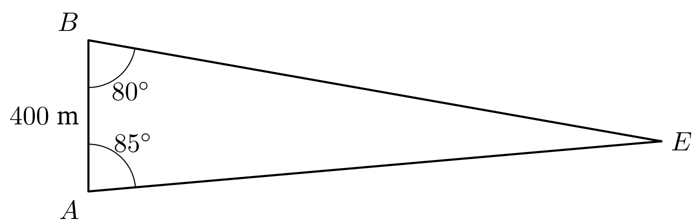
9. The angle between the two bearings is $80° - 30° = 50°$.
   By the Law of Cosines:

$$d^2 = 15^2 + 20^2 - 2(15)(20)\cos 50° = 625 - 386 = 239.$$

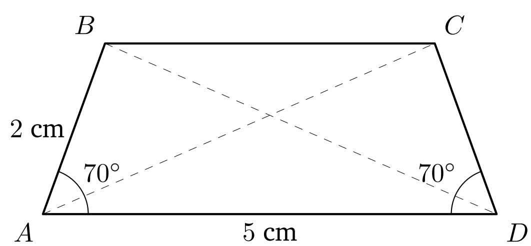
   So $d \approx 15.5$ km.

Homework

1. In a right triangle $\triangle ABC$, the legs satisfy $AB = 3\sqrt{3}$ and $BC = 9$, and AC is the hypotenuse. Find all three angles of the triangle.
2. A right triangle $\triangle ABC$ is placed so that $A = (0, 0)$, C lies on the positive x–axis, and B lies in the first quadrant. If $AB = 1$ and AB makes an angle of $35°$ with the positive x–axis, find the coordinates of point B.
3. In triangle $\triangle ABC$, let $\angle A = 40°$, $\angle B = 60°$, and $AB = 2$ cm. Find the remaining angle and all side lengths.
4. In an isosceles triangle, the angle between the equal sides is $30°$, and the height from that angle is 8. Find the lengths of all sides.
5. In the figure below, two observers are located at points A and B, separated by a distance of 400 meters. Each measures the angle to the enemy gun at point E using a special instrument. Given the angles shown in the diagram, determine how far the gun is from observer A.



6. M In the trapezoid shown below, $AD = 5$ cm, $AB = 2$ cm, and $\angle A = \angle D = 70°$. Find the length of BC and the lengths of diagonals. (Use $\sin 70° \approx 0.94$ and $\cos 70° \approx 0.34$.)



7. In triangle ABC, $b = 5$, $c = 8$, and $A = 60°$. Find side a and the area of the triangle.
8. A triangle has sides $a = 7$, $b = 8$, $c = 10$. Find all three angles.
9. In triangle ABC, $a = 5$, $b = 7$, and $\cos C = \frac{1}{7}$. Find side c and the area of the triangle.

Quick Check Answers

1. Area $= \frac{1}{2}(8)(11)\sin 65^\circ \approx 39.9$
2. Area $= \frac{1}{2}(12)(15)\sin 75^\circ \approx 86.9$
3. $5\sin^2 x + 5\cos^2 x = 5(\sin^2 x + \cos^2 x) = 5$
4. $7\sin^2 x + 4\cos^2 x = 7\sin^2 x + 4(1 - \sin^2 x) = 3\sin^2 x + 4$
5. $\sin^2 x = 1 - \cos^2 x$
6. $\sin x = \frac{4}{5}$
7. $\sin 30 = \frac{1}{2} = \cos 60$ ✓; $\quad \sin 60 = \frac{\sqrt{3}}{2} = \cos 30$ ✓
8. $\sin 25 \cdot \cos 65 + \cos 25 \cdot \sin 65 = \sin 25 \cdot \sin 25 + \cos 25 \cdot \cos 25 = \sin^2 25 + \cos^2 25 = 1$
9. $b = \dfrac{9\sin 70^\circ}{\sin 40^\circ} \approx 13.2$
10. $\sin\beta = \dfrac{10\sin 50^\circ}{14} \approx 0.547$; using a calculator, $\beta \approx 33.2^\circ$
11. $c^2 = 36 + 64 - 96\cos 45^\circ \approx 32.1$, so $c \approx 5.67$
12. Largest angle is opposite longest side. $\cos\gamma = \dfrac{16 + 25 - 36}{40} = \dfrac{1}{8}$, so $\gamma \approx 82.8^\circ$