MATH 7: HANDOUT 21

TRIGONOMETRY II: AREA, LAW OF SINES, AND LAW OF COSINES (SUMMARY)

Area of a Triangle Using Sine

Theorem 1 (Area Formula). For a triangle with sides a, b, c and opposite angles α, β, γ:

$$\text{Area} = \tfrac{1}{2}\, bc\, \sin\alpha = \tfrac{1}{2}\, ca\, \sin\beta = \tfrac{1}{2}\, ab\, \sin\gamma.$$

The angle must be the *included angle* between the two sides.

Example: Sides $b = 7$, $c = 10$, included angle $\alpha = 40°$. Area $= \frac{1}{2}(7)(10)\sin 40° \approx 22.5$.

Fundamental Trigonometric Identity

Theorem 2 (Pythagorean Identity). For any angle α: $\sin^2\alpha + \cos^2\alpha = 1$.

Consequences: $\sin^2\alpha = 1 - \cos^2\alpha$ and $\cos^2\alpha = 1 - \sin^2\alpha$.
If you know $\sin\alpha$, you can find $|\cos\alpha|$, and vice versa. The sign depends on the quadrant.
Example: If $\cos\alpha = \frac{3}{5}$ and α is acute, then $\sin\alpha = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.

Complementary Angles

For complementary angles ($\alpha + \beta = 90°$):

$$\sin(90° - \alpha) = \cos\alpha, \qquad \cos(90° - \alpha) = \sin\alpha$$

This is why cosine is called "co-sine" — the sine of the **co**mplementary angle.

The Law of Sines

Theorem 3 (Law of Sines). In any triangle with sides a, b, c opposite angles α, β, γ:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Use when: you know AAS, ASA, or SSA (ambiguous case).
Example: $\alpha = 30°$, $\beta = 45°$, $a = 12$. Then $b = \dfrac{12\sin 45°}{\sin 30°} = 12\sqrt{2}$.
Warning: When finding an angle via $\sin\beta = \ldots$, there may be two solutions (one acute, one obtuse). Check which fits.

The Law of Cosines

Theorem 4 (Law of Cosines). In any triangle:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Similarly for b^2 and c^2. Generalizes the Pythagorean theorem ($\alpha = 90° \Rightarrow \cos\alpha = 0$).

Use when: you know SAS (two sides + included angle) or SSS (all three sides).

Rearranged to find an angle: $\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

Example (find side): $b = 7$, $c = 10$, $\alpha = 60°$.

$$a^2 = 49 + 100 - 140 \cdot \tfrac{1}{2} = 79$$

So $a = \sqrt{79} \approx 8.89$.

Example (find angle): $a = 5$, $b = 7$, $c = 8$.

$$\cos C = \frac{25 + 49 - 64}{70} = \frac{10}{70} = \frac{1}{7}$$

So $C \approx 81.8°$.

Geometric insight: The term $-2bc\cos\alpha$ corrects the Pythagorean theorem:

- $\alpha < 90°$: $\cos\alpha > 0$, so $a^2 < b^2 + c^2$ (side opposite acute angle is shorter)
- $\alpha > 90°$: $\cos\alpha < 0$, so $a^2 > b^2 + c^2$ (side opposite obtuse angle is longer)

Which Law to Use?

Known information	Law	Find
AAS or ASA (two angles + side)	Sines	remaining sides
SSA (two sides + opposite angle)	Sines	other angle (ambiguous!)
SAS (two sides + included angle)	Cosines	third side
SSS (three sides)	Cosines	any angle

Example: Solving a Triangle Completely

Given $a = 8$, $b = 6$, $\gamma = 50°$. Find all remaining sides and angles.

Step 1 (LoC — find c): $c^2 = 64 + 36 - 96\cos 50° \approx 38.3$, so $c \approx 6.19$.

Step 2 (LoS — find α): $\sin\alpha = \dfrac{a\sin\gamma}{c} = \dfrac{8\sin 50°}{6.19} \approx 0.990$, so $\alpha \approx 81.9°$.

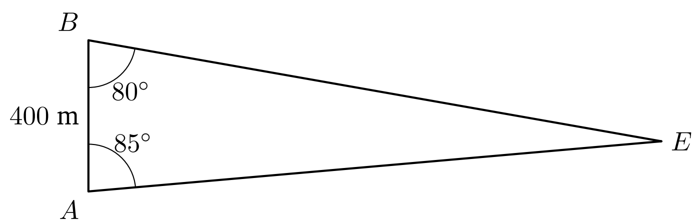
Step 3 (angle sum): $\beta = 180° - 81.9° - 50° = 48.1°$.

Common Mistakes

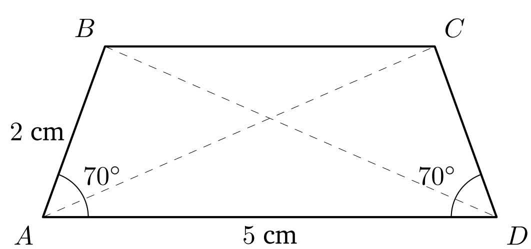
- **Wrong angle in area formula:** Must use the *included* angle between the two sides.
- **Forgetting the sign:** $\sin^2\alpha + \cos^2\alpha = 1$ gives $|\sin\alpha|$, not the sign.
- **Law of Sines ambiguity:** $\sin\beta$ can give two angles — check which fits.
- **Sign error in Law of Cosines:** It's $-2bc\cos\alpha$, not $+$.

Homework

1. In a right triangle $\triangle ABC$, the legs satisfy $AB = 3\sqrt{3}$ and $BC = 9$, and AC is the hypotenuse. Find all three angles of the triangle.
2. A right triangle $\triangle ABC$ is placed so that $A = (0, 0)$, C lies on the positive x–axis, and B lies in the first quadrant. If $AB = 1$ and AB makes an angle of $35°$ with the positive x–axis, find the coordinates of point B.
3. In triangle $\triangle ABC$, let $\angle A = 40°$, $\angle B = 60°$, and $AB = 2$ cm. Find the remaining angle and all side lengths.
4. In an isosceles triangle, the angle between the equal sides is $30°$, and the height from that angle is 8. Find the lengths of all sides.
5. In the figure below, two observers are located at points A and B, separated by a distance of 400 meters. Each measures the angle to the enemy gun at point E using a special instrument. Given the angles shown in the diagram, determine how far the gun is from observer A.



6. M In the trapezoid shown below, $AD = 5$ cm, $AB = 2$ cm, and $\angle A = \angle D = 70°$. Find the length of BC and the lengths of diagonals. (Use $\sin 70° \approx 0.94$ and $\cos 70° \approx 0.34$.)



7. In triangle ABC, $b = 5$, $c = 8$, and $A = 60°$. Find side a and the area of the triangle.
8. A triangle has sides $a = 7$, $b = 8$, $c = 10$. Find all three angles.
9. In triangle ABC, $a = 5$, $b = 7$, and $\cos C = \frac{1}{7}$. Find side c and the area of the triangle.