MATH 7: HANDOUT 20

Trigonometry I: Basic Definitions

Basic Trigonometry: Definitions

Trigonometry studies how *angles* and *side lengths* in triangles are related. We begin with the simplest case: a right triangle. Understanding sine, cosine, and tangent will let you work with heights, distances, shadows, slopes, and later with waves and periodic motion.

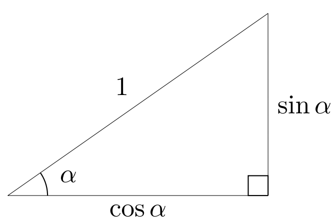
Defining Sine, Cosine, and Tangent

Consider a right triangle whose hypotenuse has length 1. Let one of the acute angles be α. The two legs are then named:

- the **opposite** side (opposite α),
- the **adjacent** side (next to α).

We *define*:

$$\sin\alpha = \text{opposite side}, \qquad \cos\alpha = \text{adjacent side}.$$



Why did we start with a triangle where the hypotenuse is exactly 1? Because it makes the definitions simple: the legs *are* the sine and cosine.

But here's the key insight: **the size of the triangle doesn't matter**.

Why? Because all right triangles with the same acute angle α are *similar*. Similar triangles have proportional sides, so the *ratios* of sides are always the same. Whether the hypotenuse is 1, 5, or 1000, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ depends only on the angle α.

This means we can define sine and cosine for *any* right triangle:

$$\sin\alpha = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \cos\alpha = \frac{\text{adjacent side}}{\text{hypotenuse}}.$$

The third basic trigonometric function is the **tangent**:

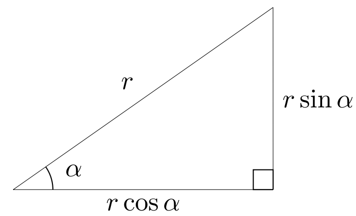
$$\tan\alpha = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sin\alpha}{\cos\alpha}.$$

Quick Check

1. In a right triangle, the side opposite angle α has length 7 and the hypotenuse has length 10. Find $\sin\alpha$ and $\cos\alpha$.
2. If $\sin\alpha = \frac{4}{5}$, find $\cos\alpha$ assuming the triangle is right.
3. Compute $\tan\alpha$ using $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ when $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.
4. In a right triangle, one leg is 5 and the hypotenuse is 13. Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ for the angle opposite the leg of length 5.

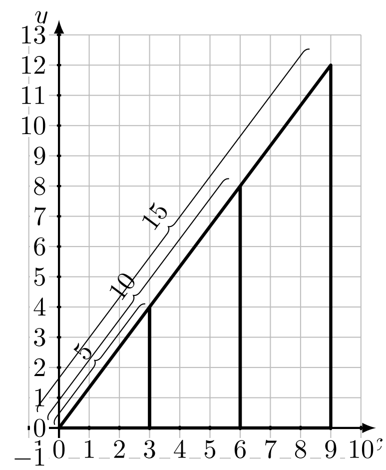
Scaling Up: Any Right Triangle

If the hypotenuse is r (instead of 1), then by similarity the legs scale proportionally:



This is useful: if you know the hypotenuse and the angle, you can immediately find the legs.

Example 1. Three Similar Triangles:



All three right triangles in the picture have the *same angle* α at the origin, but different sizes.

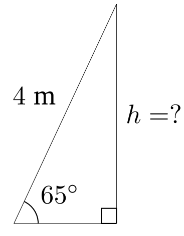
$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} = \frac{8}{10} = \frac{12}{15},$$
$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} = \frac{6}{10} = \frac{9}{15},$$
$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} = \frac{8}{6} = \frac{12}{9}.$$

No matter how big the triangle is, these ratios stay the same as long as the angle is the same.

Solving Right Triangles

If you know one side and one acute angle of a right triangle, you can find any other side. The key is choosing the right trig function.

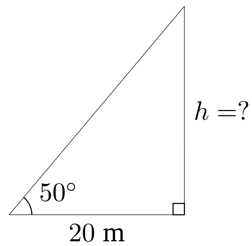
Example 2. A ladder leans against a wall at a $65°$ angle with the ground. If the ladder is 4 m long, how high up the wall does it reach?



We know the hypotenuse (4 m) and want the opposite side (h). Use sine:

$$\sin 65^\circ = \frac{h}{4} \quad \Longrightarrow \quad h = 4 \sin 65^\circ \approx 3.63 \text{ m}$$

Example 3. From a point 20 m away from the base of a tower, the angle of elevation to the top is 50°. How tall is the tower?



We know the adjacent side (20 m) and want the opposite side (h). Use tangent:

$$\tan 50^\circ = \frac{h}{20} \quad \Longrightarrow \quad h = 20 \tan 50^\circ \approx 23.8 \text{ m}$$

Which function to use? Look at what you *know* and what you *want*:

- **Opposite and Hypotenuse** $\to$ use $\sin$
- **Adjacent and Hypotenuse** $\to$ use $\cos$
- **Opposite and Adjacent** $\to$ use $\tan$

Quick Check

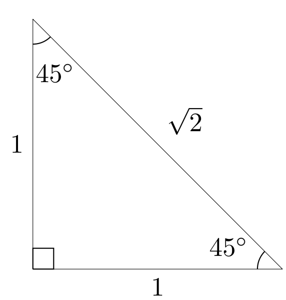
5. A right triangle has hypotenuse 12 and $\sin \alpha = \frac{2}{3}$. Find the length of the side opposite α.
6. Another right triangle has the same angle α but hypotenuse 30. What is the opposite side now? (No calculator.)
7. True or false (explain in one sentence): Changing the size of a triangle changes its trigonometric ratios.

Special Angles

There are a few angles for which the trigonometric ratios can be computed *exactly*. These values come from two very simple geometric constructions: an isosceles right triangle (for 45°) and an equilateral triangle (for 30° and 60°).

The 45° Angle (Isosceles Right Triangle)

Take a right triangle whose legs both have length 1. Then the two acute angles must be 45° each.



By the Pythagorean theorem,

$$\text{hypotenuse} = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Thus,

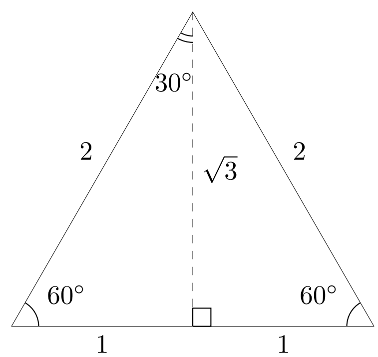
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

And

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1.$$

The 30° and 60° Angles (Equilateral Triangle)

Start with an equilateral triangle of side length 2. All its angles are 60°. Draw an altitude; it splits the triangle into two 30°–60°–90° right triangles.



In one of these right triangles:

- the hypotenuse $= 2$
- the shorter leg (adjacent to 60°) $= 1$
- the longer leg (opposite 60°) $= \sqrt{2^2 - 1^2} = \sqrt{3}$ (by Pythagoras Theorem)

Thus,

$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

And for 60°:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \tan 60^\circ = \sqrt{3}.$$

The Angle 0°

As the angle α becomes very small, the opposite side shrinks toward 0 while the adjacent side approaches the full hypotenuse.

Thus:

$$\sin 0^\circ = 0, \qquad \cos 0^\circ = 1, \qquad \tan 0^\circ = 0.$$

The Angle 90°

As the angle α opens wider and approaches a right angle, the side *opposite* the angle becomes almost as long as the hypotenuse, while the *adjacent* side becomes very small.

In the limiting case of a full right angle, the adjacent side collapses to 0 while the opposite side becomes equal to the hypotenuse.

Thus:

$$\sin 90^\circ = 1, \qquad \cos 90^\circ = 0, \qquad \tan 90^\circ \text{ is undefined.}$$

Geometrically, $\tan 90°$ is undefined because the "opposite over adjacent" ratio

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}}$$

would require dividing by 0 when the angle is exactly $90°$.

Summary Table

Angle	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin \alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \alpha$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

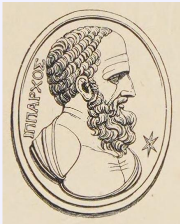
Quick Check

8. In a $45°$–$45°$–$90°$ triangle, the legs are each 5. Find the hypotenuse and compute $\sin 45°$.
9. In a $30°$–$60°$–$90°$ triangle, the side opposite $30°$ is 4. Find the hypotenuse and the side opposite $60°$.
10. Without a calculator, use the table above to find: $\sin 60°$, $\cos 30°$, $\tan 45°$.
11. Why is $\tan 90°$ undefined?
12. Which is larger: $\sin 30°$ or $\sin 60°$? Explain briefly.

Historical Note

The word "sine" comes from a long translation path. Ancient Indian mathematicians used the word *jya* ("bowstring"). Arabic scholars wrote it as *jiba*, which medieval translators misread as *jaib*, meaning "fold". The Latin equivalent was *sinus*, and from there we get the English word "sine". Cosine originally meant "the sine of the complementary angle."

Hipparchus (2nd century BCE) created the first trigonometric table, using chords of a circle. This allowed ancient astronomers to compute distances to planets and predict eclipses. Modern sine, cosine, and tangent functions grew from these early astronomical ideas.



Hipparchus (c.190–c.120BC)

Key Takeaways

- The three basic trigonometric ratios are:

$$\sin\alpha = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\alpha = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}}.$$

- **Exact vs. approximate answers:** It's often better to leave answers in exact form using trig functions (like $50\tan 35°$ or $10\sin 40°$) rather than computing decimal approximations. The exact form is more precise, and you can always use a calculator later if a decimal is needed.
- The trigonometric ratios depend *only on the angle*, not on the size of the triangle.
- The relationship $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ connects all three functions.
- For special angles, memorize: $\sin 30° = \frac{1}{2}$, $\sin 45° = \frac{\sqrt{2}}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$.
- The values of $\cos$ at these angles are the reverse order: $\cos 30° = \frac{\sqrt{3}}{2}$, $\cos 45° = \frac{\sqrt{2}}{2}$, $\cos 60° = \frac{1}{2}$.

Common Mistakes

- **Confusing opposite and adjacent:** The "opposite" side is across from the angle you are using; the "adjacent" side is next to the angle (but not the hypotenuse). Always identify which angle you are working with first.
- **Using the wrong angle:** In a right triangle with angles α and β, the side opposite α is adjacent to β. Make sure you use the correct angle in your calculation.
- **Forgetting that $\tan 90°$ is undefined:** Since $\cos 90° = 0$, the ratio $\tan 90° = \sin 90°/\cos 90°$ involves division by zero.
- **Calculator mode errors:** Make sure your calculator is in degree mode when working with degrees. Later, when we use radians, you'll need to switch modes.

Classwork

1. In a right triangle, the side opposite angle α has length 5 and the hypotenuse has length 13. Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$.
2. Without a calculator, find the exact values:
   (a) $\sin 30^\circ + \cos 60^\circ$
   (b) $\sin 45^\circ \cdot \cos 45^\circ$
   (c) $\tan 60^\circ - \tan 30^\circ$
3. A rope is attached to the top of a 15 m flagpole and makes a 60° angle with the ground. How long is the rope?
4. In a 30°–60°–90° triangle, the hypotenuse is 10. Find the lengths of both legs.
5. A ramp rises at a 20° angle. If the ramp is 8 m long, how high does it rise?
6. A ship sails 10 km due east, then turns and sails 8 km in a direction 60° north of east. How far east and how far north is it from its starting point?
7. From a point on the ground 50 m from the base of a building, the angle of elevation to the top of the building is 55°. From the same point, the angle of elevation to the top of a flagpole on the roof is 60°. How tall is the flagpole?

Classwork Solutions

1. By Pythagorean theorem, adjacent side $= \sqrt{13^2 - 5^2} = \sqrt{144} = 12$.
   $\sin\alpha = \dfrac{5}{13}, \quad \cos\alpha = \dfrac{12}{13}, \quad \tan\alpha = \dfrac{5}{12}.$
2. (a) $\sin 30^\circ + \cos 60^\circ = \frac{1}{2} + \frac{1}{2} = 1$
   (b) $\sin 45^\circ \cdot \cos 45^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$
   (c) $\tan 60^\circ - \tan 30^\circ = \sqrt{3} - \frac{1}{\sqrt{3}} = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$
3. The flagpole is opposite the 60° angle, and the rope is the hypotenuse.
   $\sin 60^\circ = \dfrac{15}{\text{rope}}$, so rope $= \dfrac{15}{\sin 60^\circ} = \dfrac{15}{\sqrt{3}/2} = \dfrac{30}{\sqrt{3}} = 10\sqrt{3} \approx 17.3$ m.
4. Short leg (opposite 30°) $= 10 \cdot \sin 30^\circ = 10 \cdot \frac{1}{2} = 5$.
   Long leg (opposite 60°) $= 10 \cdot \sin 60^\circ = 10 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3} \approx 8.66$.
5. Height $= 8 \sin 20^\circ \approx 2.74$ m.
6. First leg: 10 km east gives position $(10, 0)$.
   Second leg: 8 km at 60° north of east adds $(8\cos 60^\circ, 8\sin 60^\circ) = (4, 4\sqrt{3})$.
   Final position: $(14, 4\sqrt{3})$.
   East: 14 km. North: $4\sqrt{3} \approx 6.93$ km.
7. Let h be the building height and p be the flagpole height.
   From the angle to the building top: $\tan 55^\circ = \dfrac{h}{50}$, so $h = 50\tan 55^\circ \approx 71.4$ m.
   From the angle to the flagpole top: $\tan 60^\circ = \dfrac{h+p}{50}$, so $h + p = 50\tan 60^\circ = 50\sqrt{3} \approx 86.6$ m.
   Flagpole height: $p = 50\sqrt{3} - 50\tan 55^\circ \approx 15.2$ m.

Homework

Problems marked with **M** *or unmarked are expected from every student. Problems marked with* **H** *are optional challenge problems.*

1. Which quantity is greater?

   (a) 0 or $\sin 0°$

   (b) 1 or $\sin 30°$

   (c) $\sin 45°$ or $\cos 45°$

   (d) $\cos 60°$ or $\sin 30°$

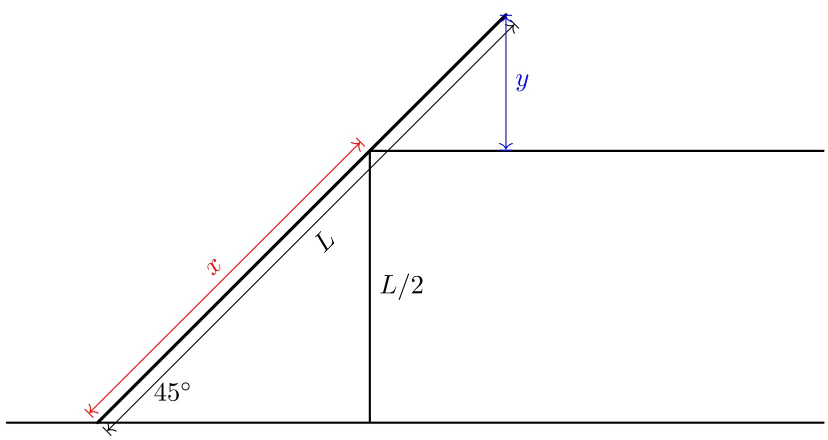
2. From the top of a 50 m lighthouse, the angle of depression to a boat is $12°$. How far is the boat from the shore?

3. A building of unknown height casts a 20 m shadow when the sun's angle of elevation is $53°$. Find the height of the building.

4. A ladder of length L rests on a ledge whose height is one half of the ladder's length. The ladder makes a $45°$ angle with the ground.

   (a) How long is the segment of the ladder between the ground and the point where it meets the ledge? *(Marked by x)*

   (b) How much higher is the top of the ladder above the ledge? *(Marked by y)*



5. A cruise ship sails 3 miles due north, then turns northwest and travels another 3 miles. How far is it from its starting point? *(Northwest is the direction that bisects the angle between north and west, i.e. $45°$ west of north.)*

6. An airplane flies 100 km due north, then turns $60°$ to the east and flies another 80 km. How far east and how far north is it from the starting point?

7. **M** Let $ABCD$ be a parallelogram with $AB = 1$, $AD = 3$, and $\angle A = 40°$. Find the lengths of both diagonals of the parallelogram.

8. **H** Prove that the area of a triangle $\triangle ABC$ can be computed using the formula

$$A = \frac{1}{2} AB \cdot AC \cdot \sin \angle A.$$

   (Hint: express the altitude from B in terms of AC and $\sin \angle A$.)

9. A regular 12-gon is inscribed in a circle of radius 1. Find the side length. (Hint: the central angle is $30°$.)

10. M What is the area of a regular pentagon inscribed in a circle of radius 10? *(Your answer should use trigonometric functions.)*

Quick Check Answers

1. $\sin\alpha = \frac{7}{10}$, $\cos\alpha = \frac{\sqrt{51}}{10}$
2. $\cos\alpha = \frac{3}{5}$
3. $\tan\alpha = \frac{3}{4}$
4. $\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$, $\tan\alpha = \frac{5}{12}$
5. 8
6. 20
7. False — the ratios depend only on the angle, not on the size.
8. Hypotenuse $= 5\sqrt{2}$, $\sin 45° = \frac{\sqrt{2}}{2}$
9. Hypotenuse $= 8$, opposite $60° = 4\sqrt{3}$
10. $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, 1
11. Adjacent side $= 0$, so division by zero
12. $\sin 60° > \sin 30°$ (larger angle $\Rightarrow$ larger opposite side)