MATH 7: HANDOUT 19

Coordinate Geometry III

Review of Quadratic Equations

Before studying graphs of quadratic functions, let us recall what we already know about quadratic expressions and equations.

- A **quadratic polynomial** is an expression of the form

$$p(x) = ax^2 + bx + c, \qquad a \neq 0.$$

- The **roots** of a quadratic polynomial are the solutions to $p(x) = 0$. If the roots are x_1 and x_2, then the polynomial factors as

$$p(x) = a(x - x_1)(x - x_2).$$

- (**Vieta's formulas**) If x_1, x_2 are the roots of $x^2 + bx + c$, then

$$x_1 + x_2 = -b, \qquad x_1x_2 = c.$$

- (**Completing the square**) Any quadratic polynomial can be rewritten as

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}, \qquad D = b^2 - 4ac.$$

- From this we obtain the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{D}}{2a}.$$

  If $D < 0$ there are no real roots; if $D \geq 0$ the roots are real.

- The completed-square form also shows us the **vertex** of the parabola:
  - If $a > 0$, the parabola opens upward and its **minimum** value is $-\dfrac{D}{4a}$, achieved at $x = -\dfrac{b}{2a}$.
  - If $a < 0$, the parabola opens downward and its **maximum** value is $-\dfrac{D}{4a}$, achieved at the same point $x = -\dfrac{b}{2a}$.

Graphs of Quadratic Functions

A **parabola** is the graph of a function of the form $y = ax^2 + bx + c$. The simplest parabola is $y = x^2$. Every other quadratic graph can be obtained from $y = x^2$ through transformations.

Vertical and Horizontal Shifts

- The graph of $y = x^2 + b$ is obtained by shifting the graph of $y = x^2$ **up** by b units (or down if $b < 0$).
- The graph of $y = (x + a)^2$ is obtained by shifting the graph of $y = x^2$ **left** by a units (or right if $a < 0$).
- Combining the two, $y = (x + a)^2 + b$ is the graph of a parabola with vertex at $(-a, b)$.

Every quadratic function $y = ax^2 + bx + c$ can be rewritten in the form

$$y = a(x - h)^2 + k$$

by completing the square. This reveals the vertex at (h, k) and whether the parabola opens up or down.

Quick Check

1. Complete the square for $y = x^2 + 4x + 1$.
2. What is the vertex of $y = (x - 3)^2 + 5$?
3. Does the parabola $y = -2(x + 1)^2 + 4$ open upward or downward?

Example 1. Graph the quadratic function $y = x^2 - 2x - 1$.

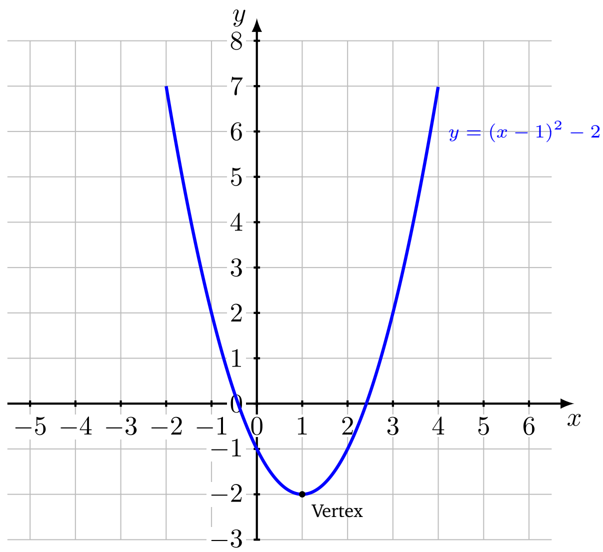
Solution. Completing the square:

$$x^2 - 2x - 1 = (x - 1)^2 - 2.$$

So the graph is the parabola $y = (x - 1)^2 - 2$, obtained from $y = x^2$ by:

- shifting **right** by 1 unit,
- shifting **down** by 2 units.

The vertex is at $(1, -2)$.



Axis of Symmetry

Every parabola is symmetric about a vertical line passing through its vertex. This line is called the **axis of symmetry**.

For a parabola in vertex form $y = a(x - h)^2 + k$, the axis of symmetry is the vertical line $x = h$.

For a parabola in standard form $y = ax^2 + bx + c$, the axis of symmetry is $x = -\dfrac{b}{2a}$.

Parabola Quick Reference

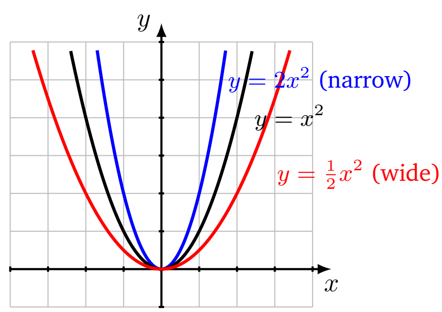
For the parabola $y = a(x-h)^2 + k$:

Vertex	(h, k)
Axis of symmetry	$x = h$
Opens upward	$a > 0$ (has minimum at vertex)
Opens downward	$a < 0$ (has maximum at vertex)
Wider than $y = x^2$	$\|a\| < 1$
Narrower than $y = x^2$	$\|a\| > 1$

For the parabola $y = ax^2 + bx + c$:

- Vertex x-coordinate: $x = -\dfrac{b}{2a}$
- Vertex y-coordinate: $y = c - \dfrac{b^2}{4a}$ (or substitute x into the equation)
- y-intercept: c (set $x = 0$)
- Discriminant $D = b^2 - 4ac$ determines the number of x-intercepts

How $|a|$ Affects Width



Larger $|a|$ makes the parabola **narrower** (steeper); smaller $|a|$ makes it **wider** (flatter).

Using Symmetry to Find Roots

Since a parabola is symmetric about its axis, if you know one x-intercept and the axis of symmetry, you can find the other x-intercept by reflection.

Example: If a parabola has axis of symmetry $x = 4$ and one x-intercept at $x = 1$, the other x-intercept is at $x = 7$ (since 1 and 7 are equidistant from 4).

Quick Check

4. Does $y = 3(x-2)^2 + 1$ have a maximum or minimum? What is its value?
5. What is the axis of symmetry of $y = -x^2 + 6x - 5$?
6. Which parabola is wider: $y = 2x^2$ or $y = \frac{1}{2}x^2$?
7. If a parabola opens downward and its vertex is at $(3, 7)$, what is the maximum value of the function?
8. Find the x-intercepts of $y = x^2 - 5x + 6$.

Geometric Meaning of a Parabola

Besides coming from quadratic equations, a parabola has a beautiful geometric description:

*A parabola is the set of all points that are equally distant from a single point (called the **focus**) and a line (called the **directrix**).*

This viewpoint is not needed to graph $y = ax^2+bx+c$, but it explains why parabolas appear everywhere in science, engineering, and nature.

Focus and Directrix Formula: For the parabola $y = ax^2$ (vertex at origin):

$$\text{Focus: } \left(0, \frac{1}{4a}\right), \qquad \text{Directrix: } y = -\frac{1}{4a}$$

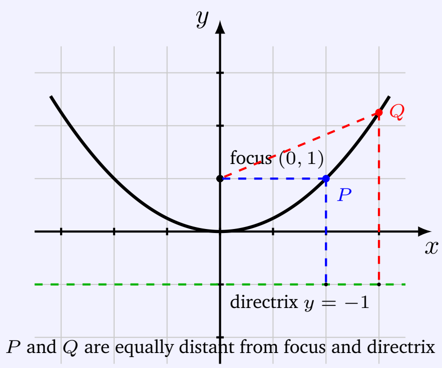
For $y = a(x-h)^2 + k$ (vertex at (h, k)), shift accordingly: focus at $\left(h, k + \frac{1}{4a}\right)$.

In the pictures below, we use the parabola

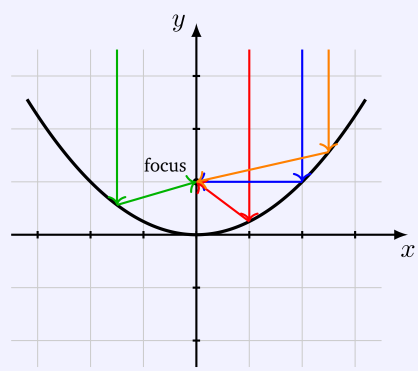
$$y = \frac{x^2}{4},$$

which has focus at $(0, 1)$ and directrix $y = -1$ (since $a = \frac{1}{4}$, we get $\frac{1}{4a} = 1$).

Focus and directrix



Reflection of rays



Why is this important? Parabolas have a special reflective property: any ray coming toward the parabola parallel to its axis bounces off the curve and passes through the **focus**. Conversely, rays coming *from* the focus reflect outward in a perfectly parallel beam.

Real-life examples.

- **Satellite dishes** are shaped as parabolas. Incoming radio waves hit the dish and reflect to a receiver placed at the focus.
- **Car headlights and flashlights** use a parabolic mirror. A small light bulb sits at the focus; the parabola reflects the light into a strong, parallel beam.
- **Solar cookers** and **solar telescopes** use giant parabolic mirrors that concentrate sunlight onto a single focus, generating extreme heat or power.
- **Suspension bridge cables** (like the Golden Gate Bridge) form a shape that is extremely close to a parabola.
- In physics, the path of a thrown ball (ignoring air resistance) is a **parabolic trajectory**.

This geometric idea links algebra, geometry, and physics: a simple quadratic formula describes shapes used to focus light, collect signals, design bridges, and model motion.

Key Takeaways

- Every quadratic function $y = ax^2 + bx + c$ can be written as $y = a(x-h)^2 + k$ by completing the square.
- The point (h, k) is the **vertex** of the parabola.
- If $a > 0$, the parabola opens upward and has a minimum at the vertex.
- If $a < 0$, the parabola opens downward and has a maximum at the vertex.
- The vertex is at $x = -\frac{b}{2a}$.
- Geometrically, a parabola is the set of points equidistant from a focus and a directrix.

Common Mistakes

- **Sign error in vertex:** If $y = (x+2)^2 - 3$, the vertex is $(-2, -3)$, not $(2, -3)$. The h in $(x-h)^2$ comes with a minus sign.
- **Forgetting the a when completing the square:** For $y = 2x^2 + 8x + 5$, first factor out the 2: $y = 2(x^2 + 4x) + 5 = 2(x+2)^2 - 8 + 5 = 2(x+2)^2 - 3$.
- **Confusing vertex with roots:** The vertex gives the maximum or minimum point, not where the parabola crosses the x-axis.
- **Direction confusion:** A *larger* value of $|a|$ makes the parabola *narrower* (steeper), not wider.

Classwork

1. Write each quadratic in vertex form and identify the vertex:
   (a) $y = x^2 - 6x + 5$
   (b) $y = x^2 + 2x - 8$
2. Sketch the graph of $y = (x-1)^2 - 4$. Label the vertex and x-intercepts.
3. A ball is thrown upward with height (in meters) given by $h(t) = -5t^2 + 20t + 1$, where t is time in seconds. Find the maximum height reached by the ball.
4. Find the equation of a parabola with vertex at $(2, -3)$ that passes through the point $(4, 5)$.

Classwork Solutions

1. (a) $y = x^2 - 6x + 5 = (x^2 - 6x + 9) - 9 + 5 = (x-3)^2 - 4$. **Vertex:** $(3, -4)$
   (b) $y = x^2 + 2x - 8 = (x^2 + 2x + 1) - 1 - 8 = (x+1)^2 - 9$. **Vertex:** $(-1, -9)$
2. For $y = (x-1)^2 - 4$:
   - Vertex: $(1, -4)$
   - For x-intercepts, set $y = 0$: $(x-1)^2 = 4 \Rightarrow x - 1 = \pm 2 \Rightarrow x = 3$ or $x = -1$.
   - x-intercepts: $(-1, 0)$ and $(3, 0)$.
3. $h(t) = -5t^2 + 20t + 1 = -5(t^2 - 4t) + 1 = -5(t^2 - 4t + 4) + 20 + 1 = -5(t-2)^2 + 21$.

   The vertex is at $(2, 21)$. Maximum height: $\boxed{21}$ meters, reached at $t = 2$ seconds.
4. Vertex form: $y = a(x-2)^2 - 3$.

   Substitute $(4, 5)$: $5 = a(4-2)^2 - 3 = 4a - 3 \Rightarrow 4a = 8 \Rightarrow a = 2$.

   Equation: $\boxed{y = 2(x-2)^2 - 3}$

Homework

Problems marked with **M** *or unmarked are expected from every student. Problems marked with* **H** *are optional challenge problems.*

1. Write each quadratic in vertex form $y = a(x-h)^2 + k$ and identify the vertex:
   (a) $y = x^2 + 8x + 12$
   (b) $y = 2x^2 - 12x + 10$
2. Sketch the graph of $y = -(x-3)^2 + 4$. Label the vertex and find the x-intercepts.
3. Find the minimum or maximum value of each function:
   (a) $f(x) = x^2 - 10x + 21$
   (b) $g(x) = -x^2 + 4x + 5$
4. Write the equation of a parabola with vertex $(3, -1)$ that passes through $(5, 7)$.
5. **M** The profit P (in dollars) from selling x items is given by $P(x) = -2x^2 + 80x - 300$. Find:
   (a) How many items should be sold to maximize profit?
   (b) What is the maximum profit?
6. **M** A projectile is launched from ground level with height $h(t) = -16t^2 + 64t$ feet after t seconds.
   (a) When does the projectile reach its maximum height?
   (b) What is the maximum height?
   (c) When does the projectile return to the ground?
7. **M** For what values of k does the equation $x^2 - 6x + k = 0$ have:
   (a) Two distinct real roots?
   (b) Exactly one real root?
   (c) No real roots?
8. **M** Find all points where the line $y = 2x - 3$ intersects the parabola $y = x^2 - 2x + 1$.
9. **H** Find all values of m for which the line $y = mx - 1$ is tangent to the parabola $y = x^2$.
   Hint: A line is tangent to a parabola when they intersect at exactly one point. Set up the intersection equation and use the discriminant.
10. **H** Prove that for any point P on the parabola $y = \frac{x^2}{4} + 1$, the distance from P to the x-axis equals the distance from P to $(0, 2)$.
    Hint: This is the focus-directrix property! The parabola has focus $(0, 2)$ and directrix $y = 0$.

Quick Check Answers

1. $y = x^2 + 4x + 1 = (x + 2)^2 - 3$
2. Vertex: $(3, 5)$
3. Opens downward (since $a = -2 < 0$)
4. Minimum of 1 (since $a = 3 > 0$, parabola opens up)
5. $x = -\dfrac{6}{2(-1)} = 3$
6. $y = \frac{1}{2}x^2$ is wider (smaller $|a|$ means wider)
7. Maximum value is 7 (the y-coordinate of the vertex)
8. $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$, so $x = 2$ and $x = 3$