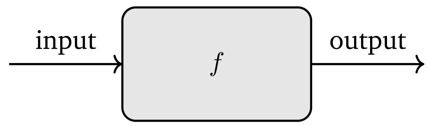
MATH 7: HANDOUT 18

COORDINATE GEOMETRY II

What is a Function?

A **function** is like a "black box" that takes an input and produces exactly one output. You put something in, the function does its work, and you get something out.



The key requirement is that *the same input always produces the same output*. If you put in the same value twice, you must get the same result both times.

Functions don't have to involve numbers! For example:

- A function could assign to each person their number of hairs.
- A function could assign to each country its capital city.
- A function could assign to each word its first letter.

In this handout, however, we focus on **numerical functions**—functions where both the input and output are numbers. We write $y = f(x)$, where x is the input, y is the output, and f is the name of the function.

Ways to Define a Function

Functions can be described in several ways:

1. By a formula: This is the most common way in algebra.

$$f(x) = x^2 + 1, \qquad g(x) = \frac{1}{x}, \qquad h(x) = |x - 3|.$$

2. By a verbal description: "$f(x)$ is the largest integer less than or equal to x."

3. By cases (piecewise): Different formulas for different inputs.

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

This is actually the absolute value function $f(x) = |x|$.

4. By a table or graph: Sometimes we only know certain input-output pairs.

Functions do not have to be "nice" or smooth. For example, consider:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

This is a perfectly valid function—every input has exactly one output—but its graph would be impossible to draw smoothly!

Key Properties of Functions

Here are some important properties we use to describe functions:

Domain. The **domain** of a function is the set of all allowed inputs. For example, $f(x) = \sqrt{x}$ has domain $x \geq 0$ because we cannot take the square root of a negative number. The function $g(x) = \frac{1}{x}$ has domain $x \neq 0$.

Range. The **range** of a function is the set of all possible outputs. For $f(x) = x^2$, the range is $y \geq 0$ because squares are never negative.

Increasing and Decreasing. A function is **increasing** if larger inputs give larger outputs: when $x_1 < x_2$, we have $f(x_1) < f(x_2)$. A function is **decreasing** if larger inputs give smaller outputs. For example, $f(x) = x^2$ is decreasing for $x < 0$ and increasing for $x > 0$.

Even and Odd. A function is **even** if $f(-x) = f(x)$ for all x (symmetric about the y-axis). A function is **odd** if $f(-x) = -f(x)$ for all x (symmetric about the origin). For example, $f(x) = x^2$ is even, while $f(x) = x^3$ is odd.

Quick Check

1. Can a function assign the same output to two different inputs? Explain.
2. What is the domain of $f(x) = \dfrac{1}{x-3}$?
3. Is $f(x) = x^3$ increasing or decreasing (or both)?
4. Write $f(x) = |x|$ as a piecewise function.
5. Is $f(x) = |x|$ even, odd, or neither?
6. What is the range of $f(x) = x^2 + 1$?

Graphs of Functions

The collection of all points (x, y) that satisfy the equation $y = f(x)$ is called the **graph** of the function. Drawing graphs helps us see how the function behaves: where it increases or decreases, how steep it is, how it curves, and how it reacts to changes in x.

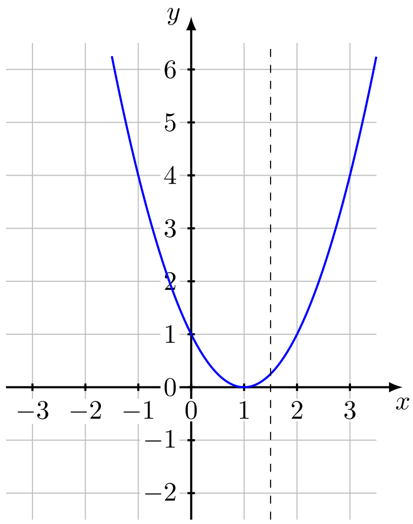
Is it a Function? (Vertical Line Test)

Not every curve in the coordinate plane is the graph of a function $y = f(x)$. For a relation to be a function, each input x must correspond to *exactly one* output y.

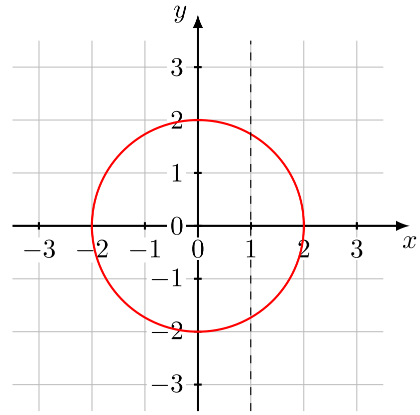
A useful visual rule is the **vertical line test**:

A graph in the xy-plane is the graph of a function $y = f(x)$ if and only if no vertical line intersects the graph in more than one point.

In the first picture below, each vertical line hits the curve at most once, so it *is* a function. In the second picture, many vertical lines intersect the circle twice, so it is *not* the graph of a function $y = f(x)$.



Graph of a function $y = f(x)$



Not a function of x

Quick Check

7. Which of the following are functions of x?
   (a) A circle given by $x^2 + y^2 = 9$.
   (b) A line given by $y = 2x - 1$.
   (c) A vertical line given by $x = 1$.
8. What is the value of $f(3)$ if $f(x) = 2x - 5$?
9. Suppose $f(1) = 7$ and $f(2) = 7$. Can f still be a function? Explain.

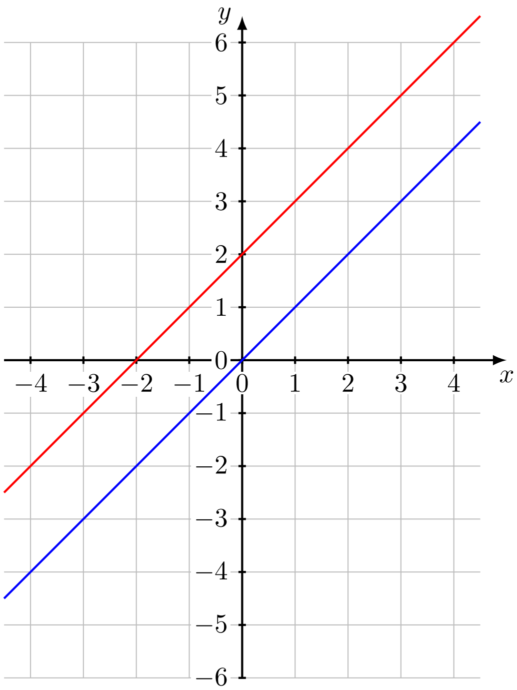
Lines

The graph of the function

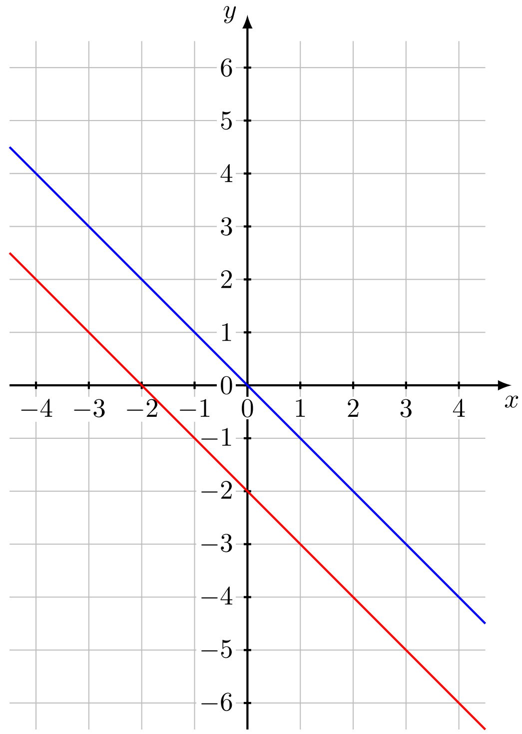
$$y = mx + b$$

is a straight line. The coefficient m is called the *slope* of the line. The number b is the y-intercept (the value of y when $x = 0$).

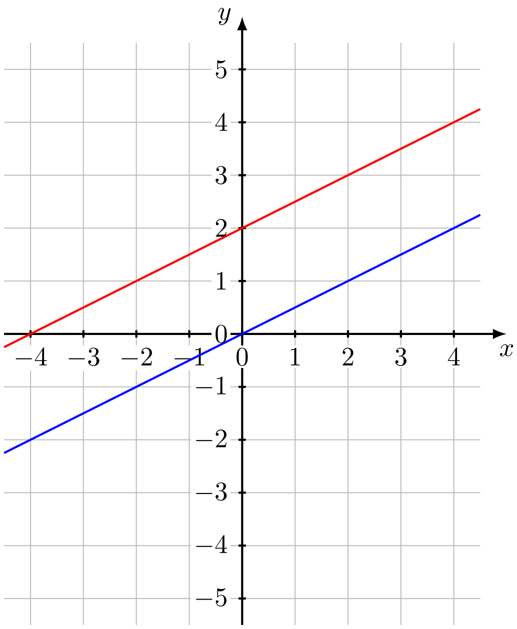
$y = x$; $y = x + 2$:



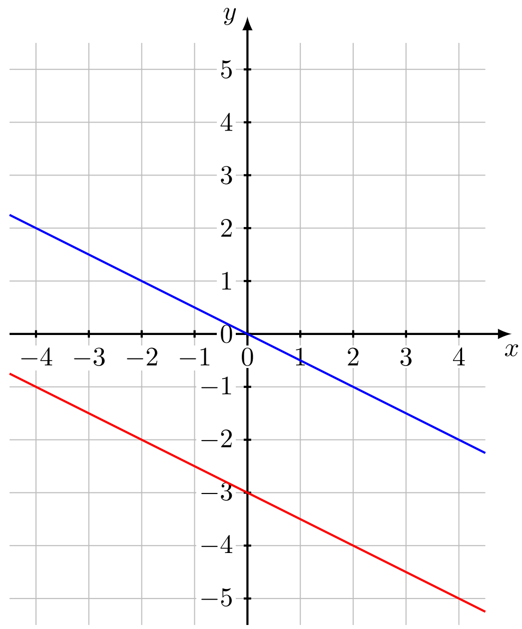
$y = -x$; $y = -x - 2$:



$y = \frac{1}{2}x$; $y = \frac{1}{2}x + 2$:



$y = -\frac{1}{2}x$; $y = -\frac{1}{2}x - 3$:

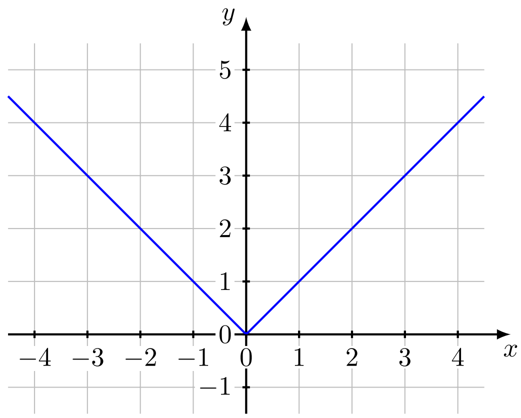


Quick Check

10. What is the slope of the line $y = -3x + 5$?
11. Which line is steeper: $y = 2x$ or $y = \frac{1}{2}x + 4$?
12. Does increasing b in $y = mx + b$ move the line up or down?
13. Write the equation of a line with slope 1 passing through $(2, 3)$.

Graph of $y = |x|$

The figure below shows the graph of the function $y = |x|$ (the absolute value of x).

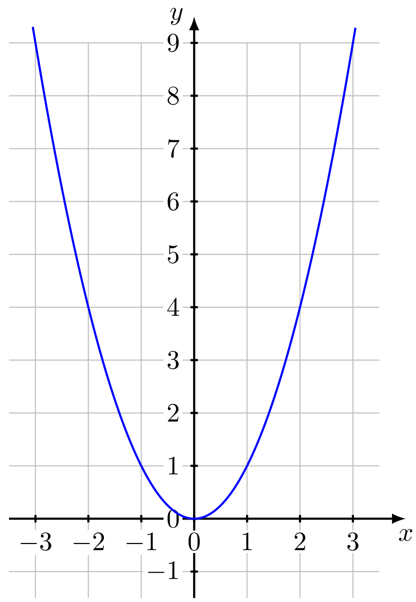


Quick Check

14. What is $|-3|$? What is $|2|$?
15. For which x is $|x|$ increasing?
16. What is the vertex (corner point) of the graph of $y = |x|$?
17. Which graph is wider: $y = |x|$ or $y = 2|x|$? Why?

Graph of $y = x^2$

The figure below shows the graph of the function $y = x^2$ (a parabola).

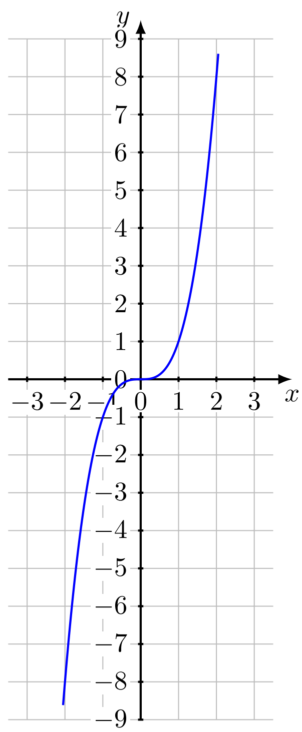


Quick Check

18. What is the vertex of $y = x^2$?
19. Is the graph symmetric about the x-axis, y-axis, both, or neither?
20. What happens to $y = x^2$ when x is doubled? What about when x is halved?
21. Which grows faster as x gets large: x^2 or x?

Graph of $y = x^3$

The figure below shows the graph of the function $y = x^3$.

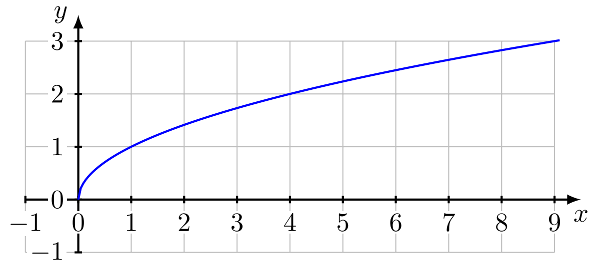


Quick Check

22. What is $(-2)^3$? What is 2^3?
23. Is $y = x^3$ increasing or decreasing? Explain.
24. Is $y = x^3$ symmetric about the y-axis, the origin, both, or neither?
25. Compare the steepness of $y = x^3$ near $x = 0$ and near $x = 2$.

Graph of $y = \sqrt{x}$

The figure below shows the graph of the function $y = \sqrt{x}$.

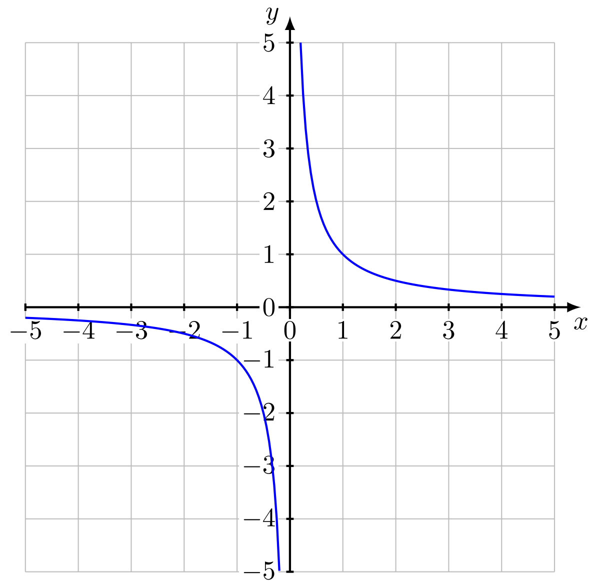


Quick Check

26. What is the domain of $y = \sqrt{x}$?
27. What is the value of $\sqrt{0}$? Of $\sqrt{9}$?
28. Does the graph continue to the left of $x = 0$? Why or why not?
29. Is $y = \sqrt{x}$ increasing or decreasing?

Graph of $y = \frac{1}{x}$

The figure below shows the graph of the function $y = \frac{1}{x}$.



An important feature of the graph of $y = \frac{1}{x}$ is that it has two **asymptotes**. An asymptote is a line that the graph gets closer and closer to, but never actually touches.

- The line $x = 0$ (the y-axis) is a **vertical asymptote**. As x approaches 0 from the left or from the right, the value of $\dfrac{1}{x}$ becomes extremely large in magnitude, so the graph shoots upward or downward close to the y-axis without crossing it.
- The line $y = 0$ (the x-axis) is a **horizontal asymptote**. As x becomes very large (positive or negative), the value of $\dfrac{1}{x}$ gets closer and closer to 0, so the graph approaches the x-axis but never reaches it.

Asymptotes help us understand the long-term behavior of a graph: they show where the graph goes when x is very large, or when x approaches a value where the function is not defined.

Quick Check

30. What is the domain of $y = \frac{1}{x}$?
31. Name the vertical asymptote and the horizontal asymptote.
32. Is $y = \frac{1}{x}$ positive or negative when x is negative?
33. What happens to $y = \frac{1}{x}$ as x becomes very large?

Parent Functions to Know

Function	Domain	Range	Key Features
$y = x$	$(-\infty, \infty)$	$(-\infty, \infty)$	Line through origin, slope 1
$y = \|x\|$	$(-\infty, \infty)$	$[0, \infty)$	V-shape, vertex at origin
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$	Parabola, vertex at origin
$y = x^3$	$(-\infty, \infty)$	$(-\infty, \infty)$	S-curve through origin
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$	Half-parabola on its side
$y = \dfrac{1}{x}$	$x \neq 0$	$y \neq 0$	Hyperbola, two branches

Transformations of Graphs

When we draw the graph of a function $y = f(x)$, we can create many new graphs by shifting, stretching, or shrinking it. These operations are called *transformations*. Each transformation changes the picture in a predictable way, and learning these rules allows us to understand new graphs quickly without recomputing every point.

There are four basic types of transformations:

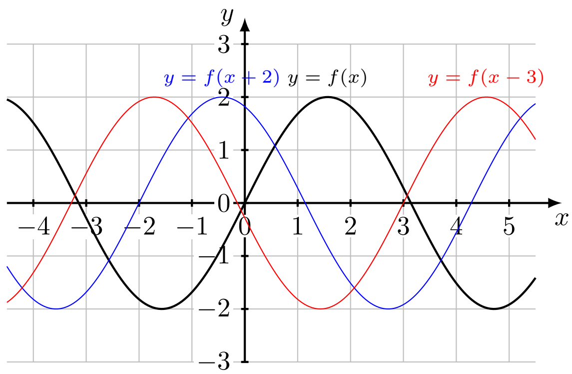
- **Horizontal translations** (shifting the graph left or right),
- **Vertical translations** (moving the graph up or down),
- **Horizontal scaling** (stretching or compressing the graph from side to side),
- **Vertical scaling** (stretching or compressing the graph vertically).

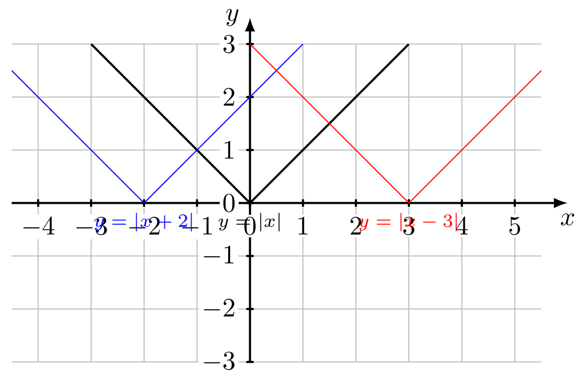
In each case we compare the original graph of $y = f(x)$ with a new graph whose equation has been modified. By examining how the equation changes, we can predict precisely how the graph will change. The examples below illustrate each type of transformation using both $y = f(x)$ (a wave-like curve; we will study $\sin x$ in a later handout) and $y = |x|$ as our base graphs.

Horizontal Translations. Replacing x with $x+a$ shifts the graph **left** by a units, and replacing x with $x-a$ shifts the graph **right** by a units. Thus

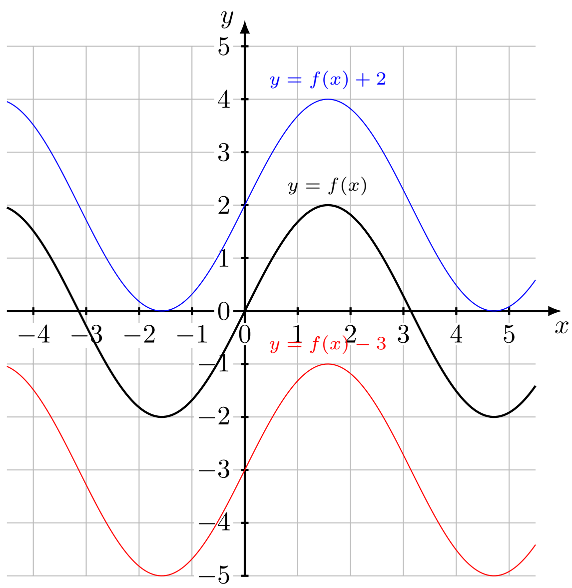
$$y = f(x+2) \text{ is the graph of } y = f(x) \text{ shifted left by } 2, \qquad y = f(x-3) \text{ is shifted right by } 3.$$

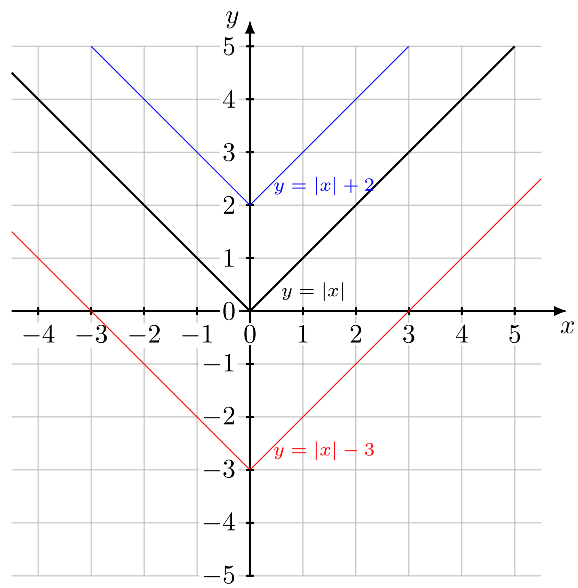
The same rules apply to $y = |x|$.



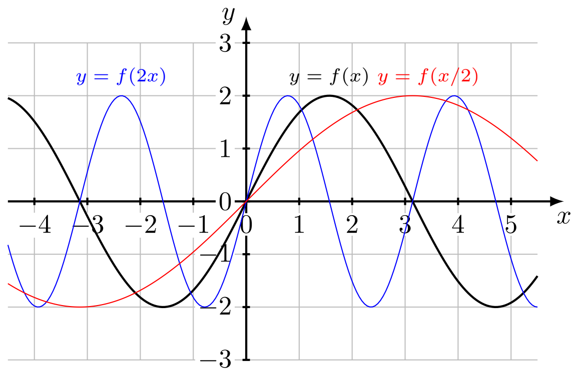


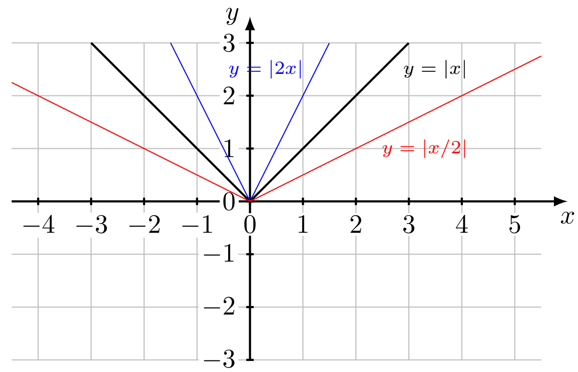
Vertical Translations. Adding a constant moves the graph up or down. The graph of $y = f(x) + a$ is the graph of $y = f(x)$ shifted **up** by a units, and $y = f(x) - a$ is shifted **down** by a units. Thus $y = f(x) + 2$ is shifted up by 2, and $y = f(x) - 3$ is shifted down by 3. The same rules apply to $y = |x|$.



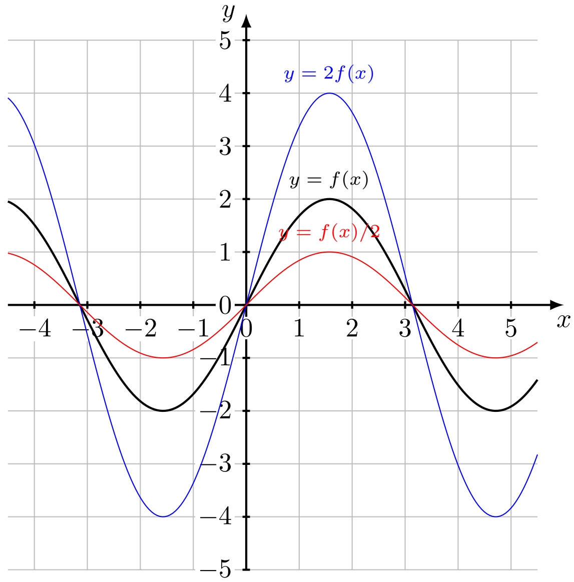


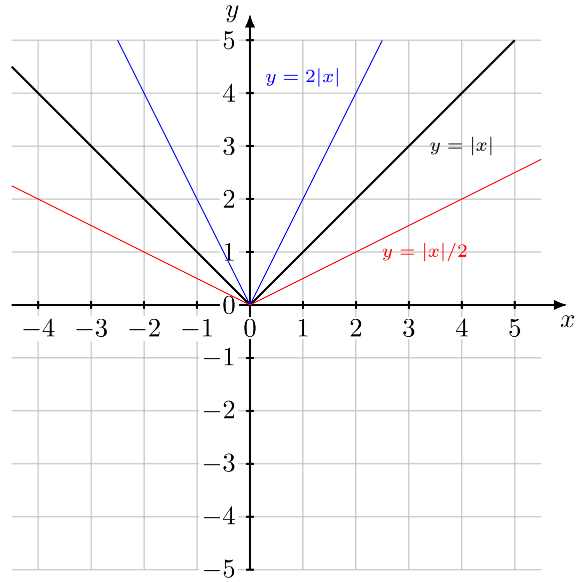
Horizontal Scaling. Replacing x with kx compresses the graph **horizontally** by a factor of k, and replacing x with x/k stretches it **horizontally** by a factor of k. Thus $y = f(2x)$ is a horizontal compression by a factor of 2, while $y = f(x/2)$ is a horizontal stretch by a factor of 2. The same rules apply to $y = |x|$.



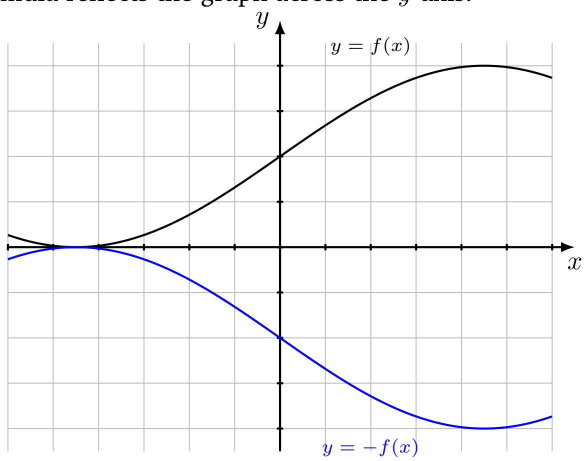


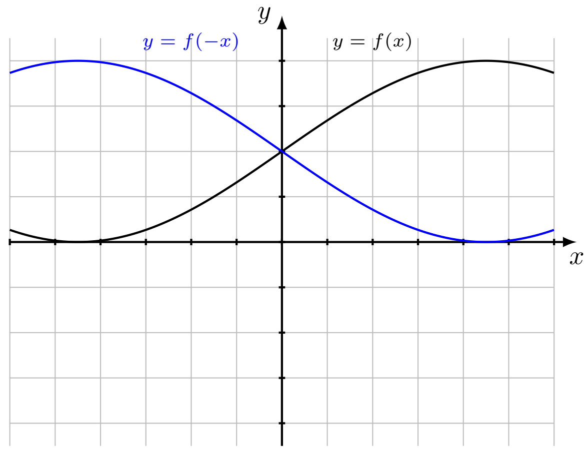
Vertical Scaling. Multiplying the function by a constant scales it vertically. The graph of $y = af(x)$ is a **vertical stretch** by a factor of a when $a > 1$, and a **vertical compression** when $0 < a < 1$. Thus $y = 2f(x)$ is stretched vertically by 2, and $y = f(x)/2$ is compressed by a factor of 2. The same rules apply to $y = |x|$.





Reflections. Multiplying a function by -1 reflects its graph across the x-axis. Replacing x by $-x$ in the formula reflects the graph across the y-axis.





Quick Check

34. How does the graph of $y = f(x) + 3$ compare to the graph of $y = f(x)$?
35. Which way does the graph of $y = f(x - 5)$ shift? By how many units?
36. Which is narrower: $y = f(2x)$ or $y = f\left(\frac{x}{2}\right)$? Why?
37. What happens to the graph when you replace $f(x)$ by $-f(x)$?
38. What transformation takes $y = f(x)$ to $y = f(-x)$?
39. Describe the sequence of transformations that turns $y = f(x)$ into
$$y = 2\,f(x + 1) - 3.$$
40. Without graphing, decide whether $y = |x - 4|$ is a shift of $y = |x|$ to the left or to the right.
41. If $y = g(x)$ is a vertical stretch of $y = f(x)$ by a factor of 3, which equation matches this transformation?
    (a) $g(x) = f(3x)$
    (b) $g(x) = 3f(x)$
    (c) $g(x) = f(x) + 3$
42. A student says that $y = f(x) - 4$ shifts the graph left by 4 units. Explain what mistake they are making.

Transformation Quick Reference

Transformation	**Equation**	**Effect**
Shift right by a	$y = f(x - a)$	All points move right
Shift left by a	$y = f(x + a)$	All points move left
Shift up by b	$y = f(x) + b$	All points move up
Shift down by b	$y = f(x) - b$	All points move down
Horizontal compress by k	$y = f(kx)$	Graph narrows ($k > 1$)
Horizontal stretch by k	$y = f(x/k)$	Graph widens ($k > 1$)
Vertical stretch by a	$y = af(x)$	Graph taller ($a > 1$)
Vertical compress by a	$y = f(x)/a$	Graph shorter ($a > 1$)
Reflect across x-axis	$y = -f(x)$	Flip vertically
Reflect across y-axis	$y = f(-x)$	Flip horizontally

Key insight: Changes *inside* the function (to x) affect the graph *horizontally* and work *opposite* to what you might expect. Changes *outside* the function affect the graph *vertically* and work as expected.

Even and Odd Functions

Some functions have special symmetry properties that make them easier to graph.

A function f is called **even** if $f(-x) = f(x)$ for all x in its domain. Even functions are symmetric about the y-axis—if you fold the graph along the y-axis, the two halves match perfectly.

Examples of even functions: $f(x) = x^2$, $f(x) = |x|$, $f(x) = x^4$.

A function f is called **odd** if $f(-x) = -f(x)$ for all x in its domain. Odd functions are symmetric about the origin—if you rotate the graph 180 around the origin, it looks the same.

Examples of odd functions: $f(x) = x$, $f(x) = x^3$, $f(x) = \frac{1}{x}$.

Quick Check

43. Is $f(x) = x^2 + 1$ even, odd, or neither? Verify by computing $f(-x)$.
44. Is $f(x) = x^3 - x$ even, odd, or neither?
45. Can a function be both even and odd? If so, give an example.

How Transformations Affect Domain and Range

When we transform a function, the domain and range may change:

- **Horizontal shifts** change the domain: $y = f(x - a)$ shifts the domain right by a.
- **Vertical shifts** change the range: $y = f(x) + b$ shifts the range up by b.
- **Horizontal scaling** scales the domain: $y = f(kx)$ compresses the domain by factor k.
- **Vertical scaling** scales the range: $y = af(x)$ stretches the range by factor a.
- **Reflections** may reverse the domain or range (e.g., $y = f(-x)$ reflects the domain).

Example: The function $y = \sqrt{x}$ has domain $[0, \infty)$ and range $[0, \infty)$.

- $y = \sqrt{x - 3}$: domain shifts to $[3, \infty)$, range stays $[0, \infty)$.
- $y = \sqrt{x} + 2$: domain stays $[0, \infty)$, range shifts to $[2, \infty)$.
- $y = 2\sqrt{x}$: domain stays $[0, \infty)$, range stretches to $[0, \infty)$ (still all non-negative, but values are doubled).

Example: Graph of $y = \dfrac{1}{x - 2} + 1$

We want to sketch the graph of the function

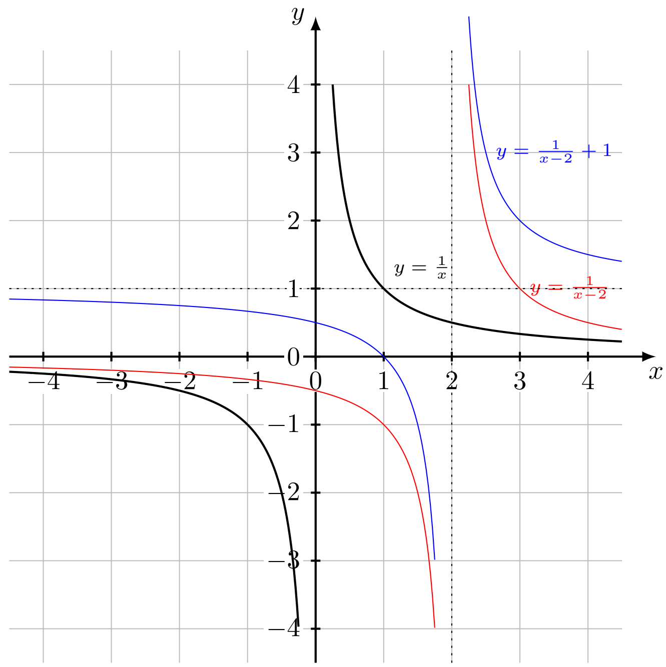
$$y = \frac{1}{x - 2} + 1.$$

Instead of starting from scratch, we begin with the basic hyperbola

$$y = \frac{1}{x}$$

and use transformations.

1. Start with the graph of $y = \dfrac{1}{x}$ (shown in black). It has vertical asymptote $x = 0$ and horizontal asymptote $y = 0$.
2. Replace x by $x - 2$ to get $y = \dfrac{1}{x - 2}$ (shown in red). This shifts the graph **2 units to the right**. Now the vertical asymptote is $x = 2$, and the horizontal asymptote is still $y = 0$.
3. Finally, add 1: the function becomes $y = \dfrac{1}{x - 2} + 1$ (shown in blue). This shifts the red graph **1 unit up**. The vertical asymptote stays at $x = 2$, and the horizontal asymptote moves up to $y = 1$.



Key Takeaways

- A function assigns exactly one output to each input; use the vertical line test to check.
- Basic graphs to know: $y = x$, $y = |x|$, $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = \frac{1}{x}$.
- **Horizontal shift:** $y = f(x - a)$ shifts right by a; $y = f(x + a)$ shifts left by a.
- **Vertical shift:** $y = f(x) + b$ shifts up by b; $y = f(x) - b$ shifts down by b.
- **Horizontal scaling:** $y = f(kx)$ compresses horizontally by factor k.
- **Vertical scaling:** $y = af(x)$ stretches vertically by factor a.
- **Reflections:** $y = -f(x)$ reflects across x-axis; $y = f(-x)$ reflects across y-axis.

Common Mistakes

- **Horizontal shift direction:** $y = f(x - 3)$ shifts *right* (not left!). The sign inside is opposite to the direction.
- **Confusing horizontal and vertical:** Replacing x with $x - a$ is horizontal; adding b to the whole function is vertical.
- **Order of transformations:** When combining transformations, horizontal operations (inside the function) happen first, then vertical operations (outside).
- **Scaling vs. shifting:** $y = 2f(x)$ is a vertical stretch (multiply outputs by 2), while $y = f(x) + 2$ is a vertical shift (add 2 to outputs).

Classwork

1. Describe the transformations that take $y = |x|$ to $y = |x - 2| + 3$, then sketch the graph.
2. The graph of $y = f(x)$ passes through the point $(1, 4)$. Find the corresponding point on:
   (a) $y = f(x) + 2$
   (b) $y = f(x - 3)$
   (c) $y = 2f(x)$
   (d) $y = f(2x)$
3. Sketch the graph of $y = -|x + 1| + 2$. Label the vertex and x-intercepts.
4. Find the equation of the function whose graph is obtained from $y = \dfrac{1}{x}$ by shifting right 3 units and down 2 units.

Classwork Solutions

1. Starting from $y = |x|$:
   - Replace x with $x - 2$: shifts right by 2 units
   - Add 3: shifts up by 3 units

   The vertex moves from $(0, 0)$ to $(2, 3)$.
2. If $(1, 4)$ is on $y = f(x)$, meaning $f(1) = 4$:
   (a) $y = f(x) + 2$: when $x = 1$, $y = f(1) + 2 = 4 + 2 = 6$. Point: $(1, 6)$
   (b) $y = f(x - 3)$: need $x - 3 = 1$, so $x = 4$. Then $y = f(1) = 4$. Point: $(4, 4)$
   (c) $y = 2f(x)$: when $x = 1$, $y = 2f(1) = 2(4) = 8$. Point: $(1, 8)$
   (d) $y = f(2x)$: need $2x = 1$, so $x = \frac{1}{2}$. Then $y = f(1) = 4$. Point: $\left(\frac{1}{2}, 4\right)$
3. For $y = -|x + 1| + 2$:
   - Start with $y = |x|$, shift left 1 ($x \to x + 1$)
   - Reflect across x-axis (multiply by -1)
   - Shift up 2

   Vertex: $(-1, 2)$ (the maximum point since the V opens downward).
   For x-intercepts, set $y = 0$: $-|x + 1| + 2 = 0 \Rightarrow |x + 1| = 2 \Rightarrow x + 1 = \pm 2$.
   So $x = 1$ or $x = -3$. **x-intercepts: $(1, 0)$ and $(-3, 0)$.**
4. Shift right 3: replace x with $x - 3$, giving $y = \dfrac{1}{x - 3}$.

   Shift down 2: subtract 2, giving $y = \dfrac{1}{x - 3} - 2$.

Homework

Problems marked with **M** *or unmarked are expected from every student. Problems marked with* **H** *are optional challenge problems.*

1. (a) Sketch the graphs of the functions $y = |x+1|$ and $y = -x + 0.25$.
   (b) Based on your graphs, how many solutions do you think the equation
   $$|x+1| = -x + 0.25$$
   has? **Note:** You are not asked to find the solutions—only to determine how many there are.

2. Describe the transformation(s) that take the first graph to the second:
   (a) $y = |x|$ to $y = |x-3|$
   (b) $y = \sqrt{x}$ to $y = \sqrt{x} - 2$

3. Sketch the graphs of the following functions:
   (a) $y = |x-5| + 1$
   (b) $y = (x-4)^2 - 1$
   (c) $y = \sqrt{5-x}$
   (d) $y = \dfrac{1}{x+2} + 1$

4. How many solutions does each equation have? Justify using graphs (no need to solve).
   (a) $|x+2| = 3$
   (b) $x^2 = x + 4$

5. **M** Let C be the circle with center $(0,1)$ and radius 2, and let ℓ be the line with slope 1 passing through the origin. Find the intersection points of the circle C and the line ℓ, and compute the distance between these two points.

6. **M** Sketch the graph of
   $$y = |x^2 - 4|.$$
   Describe where the parabola $y = x^2 - 4$ is above or below the x-axis, and how that affects the shape of the graph of $|x^2 - 4|$.

7. **M** Sketch the graphs of the following functions:
   (a) $y = \dfrac{x+2}{x+1}$ [Hint: rewrite the numerator as $(x+1)+1$.]
   (b) $y = \left|\dfrac{1}{x-1} + 1\right|$

8. **H** Prove that the set of all points P satisfying the condition
   $$\text{distance from } P \text{ to the origin} = 2 \cdot \big(\text{distance from } P \text{ to } (0,3)\big)$$
   is a circle. Find the center and the radius of this circle.

9. **H** Sketch the graphs of the following functions:
   (a) $y = |x| + |x+1|$
   (b) $y = |x-1| + |x+1|$
   (c) $|y| = x$

   Hint: For parts (a) and (b), first sketch the graphs of each individual absolute-value expression separately, and then add their values to build the graph of the sum.

Quick Check Answers

1. Yes. For example, $f(x) = x^2$ gives $f(2) = 4$ and $f(-2) = 4$. Same output, different inputs.
2. All $x \neq 3$ (we cannot divide by zero).
3. Increasing (as x increases, x^3 increases).
4. $f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$
5. Even: $|-x| = |x|$ for all x.
6. $y \geq 1$ (since $x^2 \geq 0$, we have $x^2 + 1 \geq 1$).
7. (a) Not a function (fails vertical line test). (b) Yes, a function. (c) Not a function (vertical line).
8. $f(3) = 2(3) - 5 = 1$
9. Yes. Different inputs can produce the same output.
10. -3
11. $y = 2x$ (slope 2 vs. slope $\frac{1}{2}$)
12. Up
13. $y = x + 1$
14. $|-3| = 3$; $|2| = 2$
15. For $x > 0$
16. $(0, 0)$
17. $y = |x|$ is wider; $y = 2|x|$ is narrower (steeper)
18. $(0, 0)$
19. y-axis only
20. When x is doubled, y quadruples. When x is halved, y is quartered.
21. x^2
22. $(-2)^3 = -8$; $2^3 = 8$
23. Increasing (as x increases, y increases)
24. The origin
25. Nearly flat near $x = 0$; steep near $x = 2$
26. $x \geq 0$ (or $[0, \infty)$)
27. $\sqrt{0} = 0$; $\sqrt{9} = 3$
28. No, because $\sqrt{x}$ is undefined for negative x
29. Increasing
30. All $x \neq 0$
31. Vertical: $x = 0$; Horizontal: $y = 0$
32. Negative
33. Approaches 0
34. Shifted up by 3 units

35. Right by 5 units
36. $y = f(2x)$ (horizontal compression makes it narrower)
37. Reflects across the x-axis
38. Reflects across the y-axis
39. Shift left 1, stretch vertically by factor 2, shift down 3
40. Right (the minus sign inside means opposite direction)
41. (b) $g(x) = 3f(x)$
42. The student confused vertical and horizontal shifts. $y = f(x) - 4$ shifts *down* by 4, not left.
43. Even: $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$
44. Odd: $f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$
45. Yes; $f(x) = 0$ is both even and odd.