MATH 7: HANDOUT 18

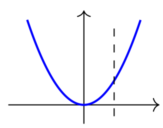
COORDINATE GEOMETRY II (SUMMARY)

Functions: Key Concepts

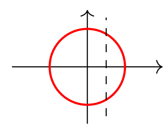
A **function** $y = f(x)$ is a rule that assigns exactly one output y to each input x. We call x the **independent variable** and y the **dependent variable**.

- **Domain:** the set of all allowed inputs. Example: $f(x) = \sqrt{x}$ has domain $x \geq 0$; $g(x) = \frac{1}{x}$ has domain $x \neq 0$.
- **Range:** the set of all possible outputs. Example: $f(x) = x^2$ has range $y \geq 0$.
- **Graph:** the set of all points (x, y) satisfying $y = f(x)$.

Vertical Line Test: A curve is the graph of a function if and only if no vertical line intersects it more than once.



Function



Not a function

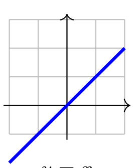
Symmetry:

- **Even functions:** $f(-x) = f(x)$ for all x. Symmetric about the y-axis. Examples: x^2, $|x|$, x^4.
- **Odd functions:** $f(-x) = -f(x)$ for all x. Symmetric about the origin. Examples: x, x^3, $\frac{1}{x}$.

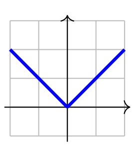
Behavior:

- **Increasing:** $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ (graph goes up as you move right).
- **Decreasing:** $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ (graph goes down as you move right).

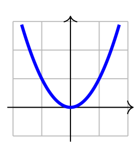
Parent Functions to Know



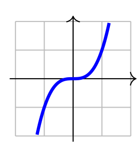
$y = x$



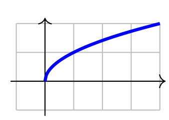
$y = |x|$



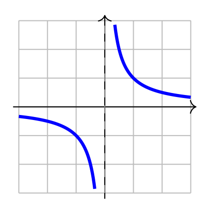
$y = x^2$



$y = x^3$



$y = \sqrt{x}$



$y = \frac{1}{x}$

Function	Domain	Range	Key Features
$y = x$	all reals	all reals	Line, slope 1, odd
$y = \|x\|$	all reals	$y \geq 0$	V-shape, vertex at origin, even
$y = x^2$	all reals	$y \geq 0$	Parabola, vertex at origin, even
$y = x^3$	all reals	all reals	S-curve, passes through origin, odd
$y = \sqrt{x}$	$x \geq 0$	$y \geq 0$	Half-parabola sideways, increasing
$y = \frac{1}{x}$	$x \neq 0$	$y \neq 0$	Hyperbola, asymptotes at axes, odd

Asymptotes: Lines that a graph approaches but never touches.

- **Vertical asymptote:** graph goes to $\pm\infty$. Example: $y = \frac{1}{x}$ has vertical asymptote $x = 0$.
- **Horizontal asymptote:** graph levels off. Example: $y = \frac{1}{x}$ has horizontal asymptote $y = 0$.

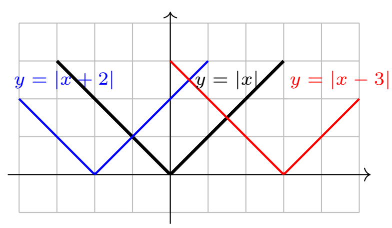
Transformations of Graphs

Starting from $y = f(x)$, we can shift, stretch, compress, or reflect the graph:

Transformation	Equation	Effect on Graph
Shift right by a	$y = f(x - a)$	All points move right by a
Shift left by a	$y = f(x + a)$	All points move left by a
Shift up by b	$y = f(x) + b$	All points move up by b
Shift down by b	$y = f(x) - b$	All points move down by b
Horizontal compress by k	$y = f(kx)$	Graph narrows (for $k > 1$)
Horizontal stretch by k	$y = f(x/k)$	Graph widens (for $k > 1$)
Vertical stretch by a	$y = af(x)$	Graph taller (for $a > 1$)
Vertical compress by a	$y = f(x)/a$	Graph shorter (for $a > 1$)
Reflect across x-axis	$y = -f(x)$	Flip vertically
Reflect across y-axis	$y = f(-x)$	Flip horizontally

Key Insight: Changes *inside* the function (to x) affect the graph *horizontally* and work *opposite* to what you might expect. Changes *outside* affect the graph *vertically* and work as expected.

Example: Horizontal shifts of $y = |x|$



How Transformations Affect Domain and Range:

- Horizontal shifts change the domain: $y = f(x - a)$ shifts domain right by a.
- Vertical shifts change the range: $y = f(x) + b$ shifts range up by b.
- Example: $y = \sqrt{x}$ has domain $[0, \infty)$, range $[0, \infty)$.
  - $y = \sqrt{x - 3}$: domain $[3, \infty)$, range $[0, \infty)$.
  - $y = \sqrt{x} + 2$: domain $[0, \infty)$, range $[2, \infty)$.

Example: Graphing $y = \dfrac{1}{x - 2} + 1$ using transformations

Start with $y = \dfrac{1}{x}$ (asymptotes at $x = 0$ and $y = 0$):

1. Replace x with $x - 2$: shift right by 2. New vertical asymptote: $x = 2$.
2. Add 1: shift up by 1. New horizontal asymptote: $y = 1$.

Vertex of Absolute Value Functions: The graph of $y = a|x - h| + k$ has vertex at (h, k).

- Opens upward if $a > 0$, downward if $a < 0$.
- Example: $y = |x - 3| + 2$ has vertex $(3, 2)$.

Homework

1. (a) Sketch the graphs of the functions $y = |x+1|$ and $y = -x + 0.25$.
   (b) Based on your graphs, how many solutions does the equation $|x+1| = -x + 0.25$ have?

2. Describe the transformation(s) that take the first graph to the second:
   (a) $y = |x|$ to $y = |x-3|$
   (b) $y = \sqrt{x}$ to $y = \sqrt{x} - 2$

3. Sketch the graphs of the following functions:
   (a) $y = |x-5| + 1$
   (b) $y = (x-4)^2 - 1$
   (c) $y = \sqrt{5-x}$
   (d) $y = \dfrac{1}{x+2} + 1$

4. How many solutions does each equation have? Justify using graphs.
   (a) $|x+2| = 3$
   (b) $x^2 = x + 4$

5. M Let C be the circle with center $(0,1)$ and radius 2, and let ℓ be the line with slope 1 passing through the origin. Find the intersection points of C and ℓ, and compute the distance between them.

6. M Sketch the graph of $y = |x^2 - 4|$. Describe where $y = x^2 - 4$ is above or below the x-axis.

7. M Sketch the graphs of the following functions:
   (a) $y = \dfrac{x+2}{x+1}$ [Hint: rewrite as $1 + \frac{1}{x+1}$.]
   (b) $y = \left| \dfrac{1}{x-1} + 1 \right|$

8. H Prove that the set of all points P satisfying

   $$\text{distance from } P \text{ to the origin} = 2 \cdot (\text{distance from } P \text{ to } (0,3))$$

   is a circle. Find the center and radius.

9. H Sketch the graphs of the following:
   (a) $y = |x| + |x+1|$
   (b) $y = |x-1| + |x+1|$
   (c) $|y| = x$