MATH 7: HANDOUT 17

Coordinate Geometry I

Coordinate Geometry: Introduction

Up to now, most of our geometry has lived in the world of rulers and compasses. But once we introduce *numbers* into geometry, a new world opens: the world of **coordinate geometry**, where every point has an address and every line has an equation.

We begin with the **coordinate plane**, consisting of:

- an **origin**, the point $(0, 0)$,
- two perpendicular number lines: the x**-axis** (horizontal) and the y**-axis** (vertical).
- Each axis carries a scale, which allows us to measure positions precisely.

Once the axes are in place, every point on the plane can be described by two numbers—its **coordinates**. To find coordinates of a point P, drop perpendiculars from P to each axis. The x-value of the foot on the x-axis is the x**-coordinate**, and the y-value of the foot on the y-axis is the y**-coordinate**. We write this as

$$P(x_0, y_0).$$

Coordinates also help us compute new points. For example, if $A(x_1, y_1)$ and $B(x_2, y_2)$ are endpoints of a segment, then the **midpoint** is

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

René Descartes (1596—1650)

In 1637, the French mathematician and philosopher **René Descartes** introduced a revolutionary idea: using numbers to describe geometry. Before Descartes, geometry and algebra lived separate lives—one dealt with shapes and diagrams, the other with equations and symbols. Descartes realized that if you place a pair of perpendicular number lines on a plane, every point can be located using just two numbers. Suddenly, a curve was not only a drawing, but also an equation.

René Descartes (1596—1650)

In his famous book *La Géométrie*, Descartes showed that geometric problems like finding intersections, constructing shapes, or describing curves could be solved using algebraic techniques. This insight gave birth to the *Cartesian coordinate system* (named after "Cartesius," the Latin form of his name). It is hard to overstate the impact of this idea: it allowed mathematicians to unify algebra and geometry into a single powerful language.

Today, everything from computer graphics and physics simulations to robotics and navigation is built on Descartes' insight—that every point in space can be given a numerical address, and every geometric figure can be described by an equation.

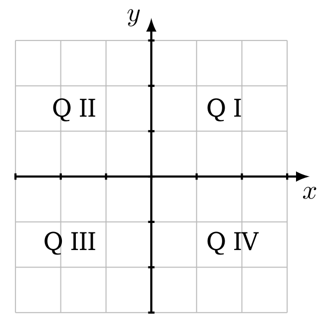
Quadrants and Symmetry

The axes divide the plane into four **quadrants**:

QI: $(+,+)$, QII: $(-,+)$, QIII: $(-,-)$, QIV: $(+,-)$.

Symmetry becomes easy to express in coordinates:

- Reflection across the x-axis: $(x,y) \to (x,-y)$
- Reflection across the y-axis: $(x,y) \to (-x,y)$
- Reflection across the origin: $(x,y) \to (-x,-y)$



Quick Check

1. In which quadrant does the point $(-3,5)$ lie?
2. What are the coordinates of the point $(4,-2)$ after reflection across the x-axis?
3. Find the midpoint of the segment from $(1,3)$ to $(5,7)$.

Lines

An equation in x and y becomes a picture when plotted on the coordinate plane. The set of all points $M(x,y)$ that satisfy an equation is called its **graph**. One of the most important families of equations is

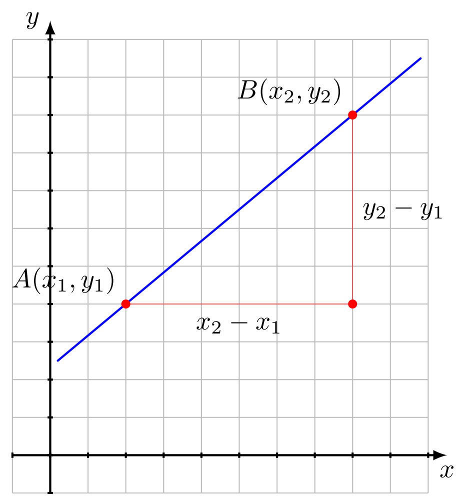
$$y = mx + b.$$

This is the equation of a **straight line**. Here:

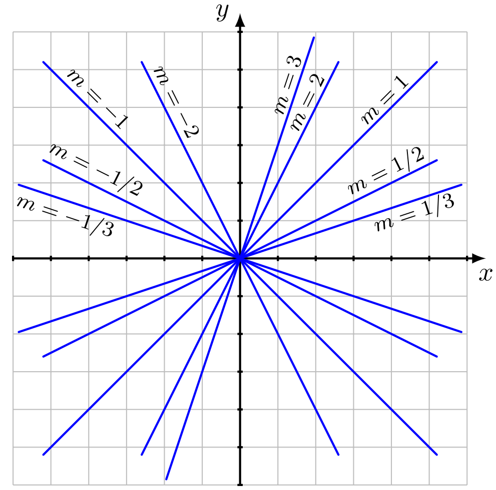
- m is the **slope**, describing steepness,
- b is the **y-intercept**, telling where the line crosses the y-axis.

Given two points $A(x_1,y_1)$ and $B(x_2,y_2)$ on the line, the slope is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$



The diagram to the right shows several lines passing through the origin, each with a different slope. Since all the lines begin at the same point, the only feature that distinguishes them is how steeply they rise or fall. A line with a larger positive slope (such as $m = 3$) climbs upward more quickly as x increases, while a smaller positive slope (such as $m = \frac{1}{2}$) rises more gently. Negative slopes tilt downward: the greater the magnitude, the steeper the descent. By seeing all of these lines together—from $m = 3$ down to $m = -2$—we can visually compare how the value of the slope affects the angle and direction of a line. In this picture, slope becomes a clear geometric idea: it is simply a measure of how much y changes when x increases by 1.
Two non-vertical lines are **parallel** exactly when they have the same slope.



Vertical and horizontal lines also have simple equations:

$$\text{vertical line: } x = k, \qquad \text{horizontal line: } y = 0x + k = k.$$

A vertical line has undefined slope; a horizontal line has slope 0.

What about **perpendicular** lines? There is an elegant relationship between their slopes, which you will discover in Homework Problem 15..

Finding Line Equations: Point-Slope Form

When you know a point (x_1, y_1) on a line and its slope m, there is a convenient form for writing its equation:

$$y - y_1 = m(x - x_1).$$

This is called the **point-slope form**. It says: the change in y from y_1 equals m times the change in x from x_1.

Example 1. Find the equation of the line through $(2, 3)$ and $(5, 9)$.

Solution. First find the slope:

$$m = \frac{9-3}{5-2} = \frac{6}{3} = 2.$$

Now use point-slope form with either point. Using $(2, 3)$:

$$y - 3 = 2(x - 2) \quad \Rightarrow \quad y = 2x - 4 + 3 = 2x - 1.$$

Quick Check

1. What is the slope of the line passing through $(1, 2)$ and $(4, 8)$?
2. Write the equation of a line with slope 3 and y-intercept -1.
3. Are the lines $y = 2x + 5$ and $y = 2x - 3$ parallel? Why or why not?

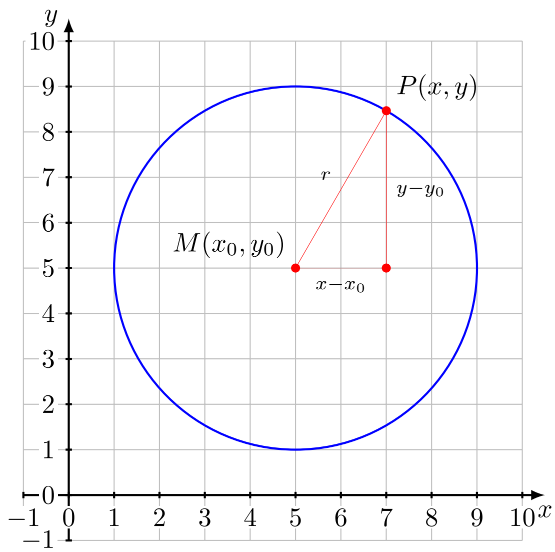
Distance Between Points. Circles

The Pythagorean theorem lets us measure how far apart two points are. For points $P(x_1, y_1)$ and $Q(x_2, y_2)$,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This distance formula immediately leads to the equation of a circle. A circle consists of all points at a fixed distance r from a center $M(x_0, y_0)$. Therefore, the equation of a circle with center $M(x_0, y_0)$ and radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$



The picture shows exactly how the circle equation comes from the distance formula.

Quick Check

1. Find the distance between the points $(1, 2)$ and $(4, 6)$.
2. Write the equation of a circle with center $(3, -1)$ and radius 5.
3. A circle has equation $(x - 2)^2 + (y + 3)^2 = 16$. What is its center and radius?

General Form of a Circle Equation

The equation $(x - x_0)^2 + (y - y_0)^2 = r^2$ is called the **standard form** of a circle. If we expand this equation, we get

$$x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 = r^2,$$

which can be rewritten as

$$x^2 + y^2 + Dx + Ey + F = 0,$$

where $D = -2x_0$, $E = -2y_0$, and $F = x_0^2 + y_0^2 - r^2$. This is called the **general form** of a circle equation.

Given an equation in general form, we can find the center and radius by **completing the square**.

Example 2. Find the center and radius of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

Solution. We group the x and y terms and complete the square for each:

$$x^2 + y^2 - 6x + 4y - 12 = 0$$
$$(x^2 - 6x) + (y^2 + 4y) = 12$$
$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$$
$$(x - 3)^2 + (y + 2)^2 = 25$$

This is a circle with center $(3, -2)$ and radius 5.

Quick Check

1. Convert $x^2 + y^2 + 2x - 8y + 8 = 0$ to standard form. What is the center and radius?
2. A circle has center $(1, -3)$ and radius 4. Write its equation in general form.

Essential Formulas

Midpoint of segment from (x_1, y_1) to (x_2, y_2):

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Slope of line through (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Distance between (x_1, y_1) and (x_2, y_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Line equations:

- Slope-intercept form: $y = mx + b$
- Point-slope form: $y - y_1 = m(x - x_1)$

Circle with center (x_0, y_0) and radius r:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Key Takeaways

- The coordinate plane uses two perpendicular axes to give every point a unique address (x, y).
- The midpoint of segment from (x_1, y_1) to (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- The slope of a line through two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$ (rise over run).
- Lines with the same slope are parallel.
- The distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The circle with center (x_0, y_0) and radius r has equation $(x - x_0)^2 + (y - y_0)^2 = r^2$ (standard form).
- The general form $x^2 + y^2 + Dx + Ey + F = 0$ can be converted to standard form by completing the square.

Common Mistakes

- **Slope order matters:** The slope formula gives the same answer whether you compute $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{y_1 - y_2}{x_1 - x_2}$, but mixing the order (like $\frac{y_2 - y_1}{x_1 - x_2}$) gives the wrong sign.
- **Circle equation signs:** In $(x - x_0)^2 + (y - y_0)^2 = r^2$, the center is (x_0, y_0), not $(-x_0, -y_0)$. If the equation is $(x + 3)^2 + (y - 2)^2 = 9$, the center is $(-3, 2)$.
- **Forgetting to square the radius:** The right side of the circle equation is r^2, not r.
- **Horizontal vs. vertical lines:** A horizontal line has equation $y = k$ (slope 0), while a vertical line has equation $x = k$ (undefined slope).

Classwork

1. Points $A(1,2)$, $B(5,2)$, and $C(5,6)$ are three vertices of a rectangle. Find the fourth vertex D and the area of the rectangle.
2. Find the equation of the line passing through the points $(2,3)$ and $(6,11)$.
3. Determine whether the triangle with vertices $A(0,0)$, $B(3,4)$, and $C(6,0)$ is isosceles, equilateral, or scalene.
4. Find all points on the x-axis that are at distance 5 from the point $(3,4)$.

Classwork Solutions

1. Since $ABCD$ is a rectangle with $A(1,2)$, $B(5,2)$, $C(5,6)$:
   - AB is horizontal (both have $y=2$), length $= 5-1 = 4$.
   - BC is vertical (both have $x=5$), length $= 6-2 = 4$.
   - D must have the same x as A and same y as C: $D = (1,6)$.
   - Area $= 4 \times 4 = \boxed{16}$.
2. First find the slope: $m = \dfrac{11-3}{6-2} = \dfrac{8}{4} = 2$.

   Using point-slope form with point $(2,3)$:

   $$y - 3 = 2(x-2) \quad \Rightarrow \quad y = 2x - 4 + 3 = 2x - 1.$$

   Answer: $\boxed{y = 2x - 1}$
3. Calculate all three side lengths:
   - $AB = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{9+16} = 5$
   - $BC = \sqrt{(6-3)^2 + (0-4)^2} = \sqrt{9+16} = 5$
   - $AC = \sqrt{(6-0)^2 + (0-0)^2} = 6$

   Since $AB = BC = 5$ but $AC = 6$, the triangle is $\boxed{\text{isosceles}}$.
4. Points on the x-axis have the form $(x,0)$. We need:

   $$\sqrt{(x-3)^2 + (0-4)^2} = 5$$

   Squaring: $(x-3)^2 + 16 = 25$, so $(x-3)^2 = 9$, giving $x - 3 = \pm 3$.

   Therefore $x = 6$ or $x = 0$.

   Answer: $\boxed{(0,0) \text{ and } (6,0)}$

Homework

Problems marked with M *or unmarked are expected from every student. Problems marked with* H *are optional challenge problems.*

1. A point B is 5 units above and 2 units to the left of point $A(7, 5)$. What are the coordinates of point B?

2. Find the coordinates of the midpoint of the segment AB, where $A(3, 11)$ and $B(7, 5)$.

3. Consider the triangle $\triangle ABC$ with vertices $A(-2, -1)$, $B(2, 0)$, and $C(2, 1)$. Find the midpoint of segment BC. Then find the length of the median from A (a median connects a vertex to the midpoint of the opposite side).

4. In this problem you will find equations that describe certain lines.

   (a) What is the equation of the y-axis?

   (b) What is the equation of the line consisting of all points that lie 5 units above the x-axis?

   (c) Is the graph of $y = x$ a line? Draw it.

   (d) Find the equation of the line that contains the points $(1, -1)$, $(2, -2)$, and $(3, -3)$.

5. A line ℓ has slope $\frac{3}{5}$. What is the slope of a line parallel to ℓ? What is the slope of a line perpendicular to ℓ?

6. Find the equation of the line through $(1, 1)$ with slope 2.

7. Find the equation of the line through the points $(1, 1)$ and $(3, 7)$. (Hint: find the slope first.)

8. (a) Find k if the point $(1, 9)$ lies on the graph of $y - 2x = k$. Then sketch the graph.

   (b) Find k if the point $(1, k)$ lies on the graph of $5x + 4y - 1 = 0$. Then sketch the graph.

9. Find the intersection point of the lines $y = x - 3$ and $y = -2x + 6$. Sketch the graphs of these lines.

10. Find the distance between the points:

    (a) $(0, 0)$ and $(3, 4)$

    (b) $(1, 2)$ and $(4, 6)$

    (c) $(-2, 3)$ and $(4, -1)$

11. Write the equation of each circle:

    (a) Center $(0, 0)$, radius 4

    (b) Center $(2, -3)$, radius 5

    (c) Center $(-1, 4)$, radius $\sqrt{7}$

12. A circle has equation $(x - 3)^2 + (y + 2)^2 = 49$. What is its center and radius?

13. M Plot the points $A(4, 1)$, $B(3, 5)$, and $C(-1, 4)$. If you plot them correctly, they will form three vertices of a square. What are the coordinates of the fourth vertex? What is the area of this square?

14. M The points $(0, 0)$, $(1, 3)$, and $(5, -2)$ are three vertices of a parallelogram. What are the coordinates of the fourth vertex? (Be careful, there might be several solutions!)

15. M For each equation below, draw the graph, then draw the line perpendicular to it that passes through the point $(0, 0)$, and finally write the equation of the perpendicular line:

(a) $y = 2x$

(b) $y = 3x$

(c) $y = -x$

(d) $y = -\frac{1}{2}x$

Based on your work, determine the general rule: if the slope of a line is k, what is the slope of the line perpendicular to it?

16. M Let l_1 be the graph of $y = x + 1$, l_2 the graph of $y = x - 1$, m_1 the graph of $y = -x + 1$, and m_2 the graph of $y = -x - 1$.

    (a) Find the intersection point of l_1 and m_1. Label this point A and write down its coordinates.

    (b) Find the intersection point of l_2 and m_2. Label this point B and write down its coordinates.

    (c) Find the midpoint of segment AB.

    (d) Let C be the intersection of l_1 with m_2, and let D be the intersection of l_2 with m_1. What type of quadrilateral is $ABCD$?

    (e) Explain why l_1 and l_2 are parallel. What is the distance between them?

17. M

    (a) Draw the graph of $x^2 + y^2 - 1 = 0$.

    (b) Draw the graph of $x^2 + (y - 1)^2 - 1 = 0$.

    (c) Draw the graph of $xy = 0$.

    (d) Draw the graph of $x^2 + y^2 = 0$.

    (e) Draw the graph of $(x^2 + y^2 - 1)(x^2 + (y - 1)^2 - 1) = 0$.

    (f) Draw the graph of $(x^2 + y^2 - 1)^2 + (x^2 + (y - 1)^2 - 1)^2 = 0$.

18. H Let A and B be points with coordinates (a, r) and (b, s), respectively. Let N be the point with coordinates $(b - a,\ s - r)$, and let O be the origin $(0, 0)$. Prove that $ON \cong AB$. (Hint: show that $ABNO$ is a parallelogram by proving that its diagonals bisect each other.)

Quick Check Answers

Quadrants and Symmetry:

1. Quadrant II (negative x, positive y)
2. $(4, 2)$
3. $(3, 5)$

Lines:

1. $m = \frac{8-2}{4-1} = \frac{6}{3} = 2$
2. $y = 3x - 1$
3. Yes, they are parallel because they have the same slope ($m = 2$).

Distance and Circles:

1. $d = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = 5$
2. $(x-3)^2 + (y+1)^2 = 25$
3. Center $(2, -3)$, radius 4

General Form of Circles:

1. $(x^2 + 2x + 1) + (y^2 - 8y + 16) = -8 + 1 + 16 = 9$, so $(x+1)^2 + (y-4)^2 = 9$. Center $(-1, 4)$, radius 3.
2. Standard form: $(x-1)^2 + (y+3)^2 = 16$. Expanding: $x^2 - 2x + 1 + y^2 + 6y + 9 = 16$, so $x^2 + y^2 - 2x + 6y - 6 = 0$.

Homework Answers

1. $B = (5, 10)$
2. Midpoint $= \left(\frac{3+7}{2}, \frac{11+5}{2}\right) = (5, 8)$
3. Midpoint of BC: $(2, 0.5)$; Median length $= \sqrt{(2-(-2))^2 + (0.5-(-1))^2} = \sqrt{16+2.25} = \sqrt{18.25} = \frac{\sqrt{73}}{2}$
4. (a) $x = 0$
   (b) $y = 5$
   (c) Yes; draw a line through the origin with slope 1
   (d) $y = -x$
5. Parallel: slope $= \frac{3}{5}$; Perpendicular: slope $= -\frac{5}{3}$
6. $y = 2x - 1$
7. Slope $= \frac{7-1}{3-1} = 3$; Equation: $y = 3x - 2$
8. (a) $k = 9 - 2(1) = 7$
   (b) $5(1) + 4k - 1 = 0 \Rightarrow k = -1$
9. Intersection: $(3, 0)$
10. Distances:
    (a) $d = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

(b) $d = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

(c) $d = \sqrt{6^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

11. Circle equations:
    (a) $x^2 + y^2 = 16$
    (b) $(x - 2)^2 + (y + 3)^2 = 25$
    (c) $(x + 1)^2 + (y - 4)^2 = 7$
12. Center $(3, -2)$, radius 7
13. Fourth vertex: $(0, 0)$; Area $= 17$ square units
14. Three solutions: $(4, -5)$, $(6, 1)$, and $(-4, 5)$
15. (a) Perpendicular line: $y = -\frac{1}{2}x$
    (b) Perpendicular line: $y = -\frac{1}{3}x$
    (c) Perpendicular line: $y = x$
    (d) Perpendicular line: $y = 2x$

    General rule: if a line has slope k, its perpendicular has slope $-\frac{1}{k}$.
16. (a) $A = (0, 1)$
    (b) $B = (0, -1)$
    (c) Midpoint of AB: $(0, 0)$
    (d) $ABCD$ is a square (or rhombus with perpendicular diagonals)
    (e) Same slope ($m = 1$); distance $= \sqrt{2}$
17. (a) Circle centered at origin with radius 1
    (b) Circle centered at $(0, 1)$ with radius 1
    (c) Two perpendicular lines: the x-axis and y-axis
    (d) Single point: $(0, 0)$
    (e) Two circles (the union of the circles from (a) and (b))
    (f) Two points: the intersections of the two circles from (a) and (b)
18. Midpoint of AN: $\left(\frac{a + (b - a)}{2}, \frac{r + (s - r)}{2}\right) = \left(\frac{b}{2}, \frac{s}{2}\right)$.

    Midpoint of BO: $\left(\frac{b + 0}{2}, \frac{s + 0}{2}\right) = \left(\frac{b}{2}, \frac{s}{2}\right)$.

    Since diagonals AN and BO have the same midpoint, $ABNO$ is a parallelogram, so $ON \cong AB$.