MATH 7: HANDOUT 16

INEQUALITIES II

Introduction

In the previous handout, we learned to solve linear, quadratic, and rational inequalities using sign analysis. We discovered that the key is to find where each factor equals zero, then determine the sign of the product on each interval.

In this handout, we extend these techniques to:

- Polynomial inequalities of higher degree
- Inequalities involving absolute values

The same fundamental principle applies: **find the roots, analyze the signs on each interval, and identify where the inequality holds**.

Solving Polynomial Inequalities

So far we have solved linear and quadratic inequalities. The same ideas extend to *all* polynomial inequalities—those involving terms like x^2, x^3, x^4, and so on.

General Method

To solve

$$p(x) \# 0, \qquad \text{where } \# \in \{<, \leq, >, \geq\},$$

we use the following steps:

- Find the real roots of the equation $p(x) = 0$ and factor the polynomial:

$$p(x) = a(x - x_1)(x - x_2) \cdots (x - x_k).$$

- The roots $x_1, x_2, \ldots, x_k$ divide the real line into intervals. On each interval, every factor has a constant sign.
- Determine the sign of each factor on every interval, and thus the sign of their product.
- If the inequality contains $\leq$ or $\geq$, include the roots themselves; if it contains $>$ or $<$, the roots are excluded.

Once everything is factored, solving the inequality becomes a sign problem.

Example 1. Solve $x^2 + x - 2 > 0$.

Solution. Factor:

$$x^2 + x - 2 = (x + 2)(x - 1).$$

The roots -2 and 1 divide the number line into three intervals:

$$(-\infty, -2), \qquad (-2, 1), \qquad (1, \infty).$$

The sign of $(x + 2)(x - 1)$ is:

- positive on $(-\infty, -2)$,
- negative on $(-2, 1)$,
- positive on $(1, \infty)$.

We want > 0, so the solution is
$$x \in (-\infty, -2) \cup (1, \infty).$$

$\square$

Example 2. Solve $-x^2 - x + 2 \geq 0$.

Solution. Factor:
$$-x^2 - x + 2 = -(x+2)(x-1).$$
This product is non-negative on the interval $[-2, 1]$ (including endpoints because of $\geq$). Thus:
$$x \in [-2, 1].$$

$\square$

Example 3. Solve $x^2 + x + 2 \geq 0$.

Solution. The discriminant is
$$\Delta = 1 - 8 = -7 < 0,$$
so the polynomial has no real roots and therefore never changes sign. Check the value at any point, say $x = 0$:
$$0^2 + 0 + 2 = 2 > 0.$$
Thus the expression is positive for all x, and the solution is
$$x \in (-\infty, +\infty).$$

$\square$

Example 4. Solve $x^2 + x + 2 < 0$.

Solution. As before, $x^2 + x + 2$ is positive for all real x. Therefore the inequality has no solutions:
$$x \in \emptyset.$$

$\square$

Example 5. Solve $x^2 - 2x + 1 > 0$.

Solution. The expression is a perfect square:
$$x^2 - 2x + 1 = (x-1)^2.$$
A square is never negative and equals zero only at $x = 1$. Thus $(x-1)^2 > 0$ for all $x \neq 1$, so the solution is:
$$x \in (-\infty, 1) \cup (1, \infty).$$

$\square$

Polynomials of Higher Degree

The same method works for any polynomial
$$x^n + bx^{n-1} + \cdots \geq 0,$$
as long as we can factor it or find its roots.

Example 6. Solve $(x+1)(x-2)^2(x-4)^3 \geq 0$.

Solution. The roots are -1, 2, and 4. These divide the number line into four intervals:

$$(-\infty, -1),\ (-1, 2),\ (2, 4),\ (4, \infty).$$

Now examine how the sign changes:

- At $x = -1$: the factor $(x + 1)$ changes sign.
- At $x = 2$: the factor $(x - 2)^2$ does *not* change sign, because it is squared.
- At $x = 4$: the factor $(x - 4)^3$ changes sign (odd power).

Starting on the far right, the expression is positive.

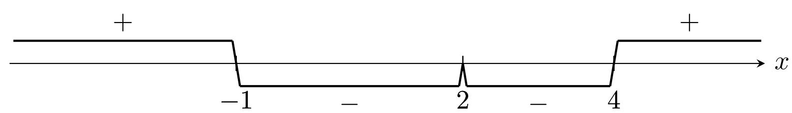
- Crossing 4 flips the sign.
- Crossing 2 does nothing (even power).
- Crossing -1 flips the sign again.

Thus the expression is:

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, 4)$	$(4, \infty)$
Sign	$+$	$-$	$-$	$+$

Since we want ≥ 0, we include intervals with a positive sign and include roots where the factor is zero and does not break the inequality:

$$x \in (-\infty, -1] \cup \{2\} \cup [4, \infty).$$



□

This method is known as the **snake method**: as you move left to right, the sign "snakes" up and down, flipping each time you pass a root of odd multiplicity.

The Multiplicity Rule

When a factor $(x - r)^k$ appears in a polynomial:

- If k is **odd** (1, 3, 5, ...): the sign **changes** at $x = r$
- If k is **even** (2, 4, 6, ...): the sign **does not change** at $x = r$

Think of it this way: an even power "bounces" off the x-axis, while an odd power "crosses" it.

Quick Check

Solve the following polynomial inequalities:

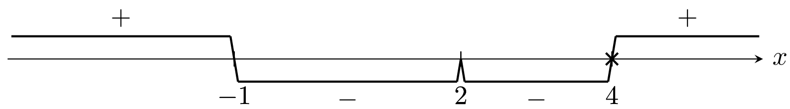
1. $(x - 1)(x + 3) > 0$
2. $x(x - 2)^2 \leq 0$
3. $(x + 1)^2(x - 3) < 0$

Example 7. Solve $\dfrac{(x + 1)(x - 2)^2}{(x - 4)^3} \geq 0.$

Solution. The same factors appear as above, except that $x = 4$ is now in the denominator. Everything is the same as the previous example, except that $x = 4$ **cannot** be included in the solution.

Thus:

$$x \in (-\infty, -1] \cup \{2\} \cup (4, \infty).$$



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Inequalities with Absolute Value

To solve inequalities containing an absolute value, we split into cases based on the definition:

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

This turns an absolute value inequality into two regular inequalities.

Geometric Interpretation

The expression $|x - a|$ represents the **distance** from x to a on the number line.

- $|x - a| < c$ means "x is *within* distance c of a" — so $x \in (a - c, a + c)$
- $|x - a| > c$ means "x is *farther than* distance c from a" — so $x \in (-\infty, a - c) \cup (a + c, \infty)$

This geometric view makes absolute value inequalities intuitive!

Example 8. Solve $|x - 4| < 7$.

Solution. We consider two cases depending on the sign of $x - 4$.

Case 1: $x - 4 \geq 0$ (that is, $x \geq 4$). Then $|x - 4| = x - 4$, so

$$x - 4 < 7 \quad \Rightarrow \quad x < 11.$$

Combine with $x \geq 4$ to get:

$$x \in [4, 11).$$

Case 2: $x - 4 < 0$ (that is, $x < 4$). Then $|x - 4| = 4 - x$, so

$$4 - x < 7 \quad \Rightarrow \quad x > -3.$$

Combine with $x < 4$:

$$x \in (-3, 4).$$

Combine the two cases:

$$x \in (-3, 11).$$

□

Shortcut for simple absolute value inequalities:

- $|A| < c$ (where $c > 0$) is equivalent to $-c < A < c$
- $|A| > c$ (where $c > 0$) is equivalent to $A < -c$ or $A > c$

Example 9. Solve $|x| \geq 3$.

Solution. Using the shortcut: $|x| \geq 3$ means $x \leq -3$ or $x \geq 3$.
Solution: $(-\infty, -3] \cup [3, \infty)$. ☐

Example 10. Solve $|2x + 1| < 5$.

Solution. Using the shortcut: $|2x + 1| < 5$ means $-5 < 2x + 1 < 5$.
Subtract 1: $-6 < 2x < 4$.
Divide by 2: $-3 < x < 2$.
Solution: $(-3, 2)$. ☐

Example 11. Solve $|3 - x| \geq 2$.

Solution. Using the shortcut: $|3 - x| \geq 2$ means $3 - x \leq -2$ or $3 - x \geq 2$.
From $3 - x \leq -2$: $-x \leq -5$, so $x \geq 5$.
From $3 - x \geq 2$: $-x \geq -1$, so $x \leq 1$.
Solution: $(-\infty, 1] \cup [5, \infty)$. ☐

Quick Check

Solve the following absolute value inequalities:

4. $|x + 2| < 5$
5. $|2x - 3| \geq 7$

Key Takeaways

- **Polynomial inequalities**: Factor completely, find all roots, then use the snake method to determine signs.
- **Even vs. odd powers**: A factor $(x - r)^k$ changes sign at r only if k is odd.
- **Rational inequalities**: Treat numerator zeros and denominator zeros separately; exclude denominator zeros from the solution.
- **Absolute value inequalities**: Split into cases based on where the expression inside changes sign.
- **Snake method**: Start from the far right (or left), determine the sign, then flip at each root of odd multiplicity.

Quick Reference

Type	Method
Polynomial	Factor completely; snake method at roots
Even power $(x - r)^{2k}$	Does NOT change sign at r
Odd power $(x - r)^{2k+1}$	Changes sign at r
Rational	Same as polynomial, but exclude denominator zeros
$\|A\| < c$	Equivalent to $-c < A < c$
$\|A\| > c$	Equivalent to $A < -c$ or $A > c$

Common Mistakes

- **Forgetting that even powers don't change sign**: $(x-2)^2$ is always ≥ 0.
- **Including denominator zeros**: In $\frac{p(x)}{q(x)} \geq 0$, points where $q(x) = 0$ must be excluded.
- **Splitting absolute value incorrectly**: Make sure to consider both $|A| = A$ when $A \geq 0$ and $|A| = -A$ when $A < 0$.
- **Forgetting to combine intervals**: After solving cases, union the results correctly.

Classwork

1. Solve the polynomial inequality: $(x-1)(x+2)(x-3) > 0$
2. Solve the rational inequality: $\dfrac{x-2}{(x+1)^2} \leq 0$
3. Solve the absolute value inequality: $|x-5| \leq 3$
4. Solve: $x^2(x-4) \geq 0$

Classwork Solutions

1. $(x-1)(x+2)(x-3) > 0$

   Roots: -2, 1, 3. Using the snake method (positive on far right):

Interval	$(-\infty,-2)$	$(-2,1)$	$(1,3)$	$(3,\infty)$
Sign	$-$	$+$	$-$	$+$

   Solution: $(-2,1) \cup (3,\infty)$

2. $\dfrac{x-2}{(x+1)^2} \leq 0$

   Numerator zero at $x=2$. Denominator zero at $x=-1$ (excluded). Since $(x+1)^2 > 0$ for $x \neq -1$, the sign depends only on the numerator.

   The fraction is ≤ 0 when $x-2 \leq 0$, i.e., $x \leq 2$, excluding $x=-1$.

   Solution: $(-\infty,-1) \cup (-1,2]$

3. $|x-5| \leq 3$

   Case 1: $x \geq 5$: $x-5 \leq 3 \Rightarrow x \leq 8$. Combined: $5 \leq x \leq 8$.

   Case 2: $x < 5$: $-(x-5) \leq 3 \Rightarrow 5-x \leq 3 \Rightarrow x \geq 2$. Combined: $2 \leq x < 5$.

   Solution: $[2,8]$

4. $x^2(x-4) \geq 0$

   Roots: 0 (multiplicity 2, even), 4 (multiplicity 1, odd). Since $x^2 \geq 0$ always, the product's sign depends only on $(x-4)$.

   The product is non-negative when $x-4 \geq 0$ (i.e., $x \geq 4$), or when $x^2 = 0$ (i.e., $x=0$).

   Solution: $\{0\} \cup [4,\infty)$

Homework

1. Solve the following polynomial inequalities.
   (a) $(x-3)(x+1)>0$
   (b) $(x+4)(x-2)\leq 0$
   (c) $x(x-5)\geq 0$
   (d) $(x-1)(x+2)(x-4)<0$

2. Solve the following absolute value inequalities.
   (a) $|x-4|<3$
   (b) $|x+2|\leq 5$
   (c) $|2x-1|>3$
   (d) $|3-x|\geq 1$

3. Solve the following absolute value equations.
   (a) $|x-3|=5$
   (b) $|2x-1|=7$
   (c) $|x^2-5|=4$

4. M Solve the following equations.
   (a) $\dfrac{x+1}{x-1}=3$
   (b) $\dfrac{x^2-9}{x+1}=x+3$
   (c) $x-\dfrac{3}{x}=\dfrac{5}{x}-x$

5. M Solve the following inequalities. Show the solution on the real line and write the final answer in interval notation.
   (a) $|x-2|>3$
   (b) $|x-1|>x+3$
   (c) $\dfrac{x-2}{x+3}\leq 3$

6. M Solve the following inequalities using the snake method. Show the solution on the real line and write the final answer in interval notation.
   (a) $(x-1)(x+2)>0$
   (b) $(x+3)(x-2)^2\leq 0$
   (c) $x(x-1)(x+2)\geq 0$
   (d) $x^2(x+1)^5(x+2)^3>0$

7. M Solve the following rational inequalities. Show the solution on the real line and write the final answer in interval notation.
   (a) $\dfrac{x+3}{x-1}>2$
   (b) $\dfrac{2x-5}{x+4}\leq 1$
   (c) $\dfrac{x^2-4}{x-3}\geq 0$
   (d) $\dfrac{x(x-1)^2}{(x+1)^2}\geq 0$

8. H Solve the following inequalities using the snake method. Show the solution on the real line and write the final answer in interval notation.
   (a) $|x^2-x|\geq 2x$

(b) $\dfrac{|x-1|}{x+2} \geq 1$

9. H **Optional:** Solve the following challenging inequalities. Show the solution on the real line and give the final answer in interval notation.

(a) $|x^2 - 4x + 3| \leq |x - 3|$

(b) $\left|\dfrac{x+2}{x-1}\right| > 2$

(c) $\dfrac{x^2 - 5x + 6}{x - 4} < x - 3$

(d) $(x-2)(x+1)(x^2 - 4x + 1) \geq 0$

(e) $\dfrac{|x-2|}{x(x-3)} \geq 1$

(f) $\sqrt{x^2 - 6x + 10} \leq x - 1$

(g) $\dfrac{1}{x-2} < \dfrac{1}{x+1}$

(h) $\dfrac{1}{x} - \dfrac{1}{x-3} \geq 1$

(i) $x^4 - 6x^2 + 5 \geq 0$

(j) $|\,|x-4| - 3\,| < 2$

Quick Check Answers

Polynomial Inequalities:

1. $(x-1)(x+3) > 0$: Roots at -3 and 1. Positive outside. Solution: $(-\infty, -3) \cup (1, \infty)$
2. $x(x-2)^2 \leq 0$: Root $x = 0$ (odd), root $x = 2$ (even, no sign change). Sign is negative for $x < 0$, zero at $x = 0$ and $x = 2$. Solution: $(-\infty, 0] \cup \{2\}$
3. $(x+1)^2(x-3) < 0$: $(x+1)^2 \geq 0$ always, so sign depends on $(x-3)$. Negative when $x < 3$, but not at $x = -1$ where it equals zero. Solution: $(-\infty, -1) \cup (-1, 3)$

Absolute Value Inequalities:

4. $|x+2| < 5 \Rightarrow -5 < x+2 < 5 \Rightarrow -7 < x < 3$. Solution: $(-7, 3)$
5. $|2x-3| \geq 7 \Rightarrow 2x-3 \geq 7$ or $2x-3 \leq -7$. So $x \geq 5$ or $x \leq -2$. Solution: $(-\infty, -2] \cup [5, \infty)$