MATH 7: HANDOUT 15

Inequalities I

Inequalities: When Things Are Not Equal

Up until now, we studied equations — perfect balances where both sides are exactly the same. But real life is rarely perfectly balanced, and in mathematics we also compare quantities using:

$$<, \quad \leq, \quad >, \quad \geq .$$

Inequalities compare two expressions. Solving an inequality means finding *all values of x* for which the comparison is true.

Just like with equations, you may:

- add or subtract the same number from both sides;
- multiply or divide by the same **positive** number.

But inequalities have one special rule: **When you multiply or divide an inequality by a negative number, the inequality sign must *flip direction*.**

This is the most common source of mistakes, so always watch the sign.

Linear Inequalities

A **linear inequality** is very similar to a linear equation, except instead of an equals sign we use

$$<, \quad \leq, \quad >, \quad \geq .$$

The goal is the same as with equations: isolate x to find all values that make the inequality true.

Fun fact

The first known use of inequality symbols "<" and ">" was by Thomas Harriot in 1631. Before that, mathematicians used long Latin phrases like "maior quam" and "minor quam". Imagine solving homework like that!

Basic Principles

Linear inequalities behave almost exactly like linear equations. You may:

- add or subtract the same number on both sides,
- add or subtract the same expression,
- multiply or divide both sides by a **positive** number without changing anything.

There is exactly one operation that changes the inequality:

Multiplying or dividing both sides of an inequality by a **negative** number reverses the direction of the inequality sign.

$$a < b \quad \Longrightarrow \quad -a > -b.$$

This "flip" is the only special rule that makes inequalities different from equations.

Strategy for Solving Linear Inequalities

To solve a linear inequality:

1. Move all x-terms to one side and constants to the other side (just like in linear equations).
2. Simplify the inequality.
3. If you divide by a negative coefficient, remember to reverse the inequality sign.
4. Express the final answer in interval notation.

Interpreting the Solution

The solution to a linear inequality is almost always an entire interval of numbers, not just a single value. To describe intervals succinctly, we use:

- parentheses ( ) for endpoints that are not included,
- brackets [ ] for endpoints that are included.

Round parentheses ( ) exclude the endpoint. Square brackets [ ] include the endpoint.
Endpoints may be included or excluded independently, so intervals like $(a, b]$ and $[a, b)$ are perfectly valid.

Examples

Example 1. Solve $x + 2 \leq 5$.

Solution. Subtract 2:

$$x \leq 3.$$

So the solution is the interval $(-\infty, 3]$.

Example 2. Solve $5 - 3x > 23$.

Solution. Subtract 5:

$$-3x > 18.$$

Divide by -3 and flip the sign:

$$x < -6.$$

Solution: $(-\infty, -6)$.

Quick Check

Solve the following linear inequalities:

1. $3x - 7 < 5$
2. $4 - 2x \geq 10$
3. $2(x + 1) > 3x - 4$

Products of Two Expressions

Many important inequalities involve products of expressions, such as

$$(x - 1)(x + 2), \qquad (x + 5)(2x - 7), \qquad \text{or even } x(x - 3).$$

Whenever this happens, the inequality becomes a question about *signs*: When is the product positive? When is it negative? When is it zero?

How to Think About Product Inequalities

A product changes sign only when one of its factors changes sign. Each factor changes sign at its root, so these roots break the number line into intervals. Inside each interval, every factor keeps a constant sign, making the product sign easy to determine.

A product of two expressions is:

- **positive** when the factors have the *same* sign,
- **negative** when the factors have *opposite* signs,
- **zero** when at least one factor is zero.

Therefore, inequalities involving a product reduce to understanding sign patterns on intervals determined by the roots.

Examples: Solving a Product Inequality

Example 3. Solve $x^2 < 4$.

Solution. First bring everything to one side:

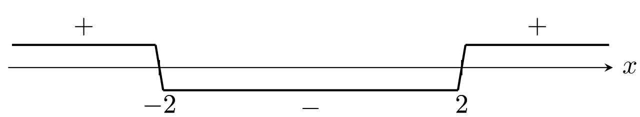
$$x^2 - 4 < 0.$$

Now factor the left-hand side:

$$(x-2)(x+2) < 0.$$

We now have a product that is required to be *negative*, which happens exactly when the two factors have *opposite* signs.

Case 1: $x - 2 > 0$ and $x + 2 < 0$. This would require $x > 2$ and $x < -2$ at the same time — impossible.
Case 2: $x - 2 < 0$ and $x + 2 > 0$. This gives $x < 2$ and $x > -2$, so $-2 < x < 2$.



Thus, the solution is the open interval $(-2, 2)$.

Example 4. Solve $(x-1)(x+2) \geq 0$.

Solution. We want the product to be non-negative. That happens when both factors are ≥ 0 or both are ≤ 0.

Case 1: Both factors non-negative. $x - 1 \geq 0$ and $x + 2 \geq 0$ gives $x \geq 1$.
Case 2: Both factors non-positive. $x - 1 \leq 0$ and $x + 2 \leq 0$ gives $x \leq -2$.

Final solution: $(-\infty, -2] \cup [1, \infty)$.

Turning Inequalities Into Products

Quadratic inequalities such as

$$x^2 < 4$$

can be rewritten as inequalities involving a product. First bring everything to one side:

$$x^2 - 4 < 0.$$

Then factor:

$$(x-2)(x+2) < 0.$$

Now the inequality becomes a product inequality, and we can analyze signs instead of squares. This method works for any quadratic that can be factored.

Quadratic Inequalities

Every quadratic inequality can be written in the form

$$ax^2 + bx + c \# 0,$$

where $a \neq 0$ and $\#$ is one of $<, \leq, >, \geq$. If the quadratic factors as

$$ax^2 + bx + c = a(x - r_1)(x - r_2),$$

then the roots r_1 and r_2 divide the number line into intervals where the signs of all factors remain constant.

To solve a quadratic inequality:

1. Rewrite it as $ax^2 + bx + c \# 0$.
2. Factor it (if possible) into $a(x - r_1)(x - r_2)$.
3. Mark the roots r_1 and r_2 on the number line.
4. Determine the sign of each factor on each interval.
5. Choose intervals satisfying the inequality.

How the Sign of a Affects the Result

The constant a is one of the factors:

$$a(x - r_1)(x - r_2).$$

Since a never changes sign:

- If $a > 0$, the sign of the product is the *same* as the sign of $(x - r_1)(x - r_2)$.
- If $a < 0$, the sign of the product is the *opposite* of the sign of $(x - r_1)(x - r_2)$.

Thus the cases $a > 0$ and $a < 0$ yield opposite solution intervals.

A Systematic Method: Sign Flips at Roots

One way to find the sign on each interval is to plug in a test point. However, there is a more systematic approach that avoids random choices.

Each factor $(x - r_i)$ changes sign only at $x = r_i$. As you pass through r_i from left to right, that factor's sign flips.

The sign of the whole product flips at each root. Once you know the sign on one interval, you can fill in the rest just by flipping signs as you cross each root.

Procedure:

1. Find the sign of the expression on one interval (usually to the far left).
2. Move left to right; at each root, flip the sign.
3. Continue until all intervals are filled in.

This avoids guessing test points and leads naturally to more advanced techniques covered later.

Special Case: Perfect Squares (Repeated Roots)

Sometimes the quadratic has a repeated root:

$$x^2 - 6x + 9 = (x - 3)^2.$$

Here the root $x = 3$ appears *twice*. Because $(x - 3)$ is squared, the sign changes of the factor cancel out.

A perfect square $(x - r)^2$ is never negative and **does not change sign** when you cross $x = r$.

Therefore:

- $(x - r)^2 > 0$ is true for all $x \neq r$,
- $(x - r)^2 \geq 0$ is true for all real x,
- $(x - r)^2 = 0$ has the single solution $x = r$,
- $(x - r)^2 < 0$ has no solutions.

More generally, for $a(x - r)^2$:

- If $a > 0$, the expression is always ≥ 0.
- If $a < 0$, the expression is always ≤ 0.

Quadratics With No Real Roots

If the discriminant $b^2 - 4ac < 0$, the quadratic has no real roots. Then it never changes sign, because it never equals zero.

To determine the sign, evaluate the quadratic at any convenient x (for example, $x = 0$). If the value is positive, it is positive for all x; if negative, negative for all x.

Thus:

- $ax^2 + bx + c < 0$ may have either no solutions or all real solutions.
- Similarly for > 0, ≤ 0, and ≥ 0.

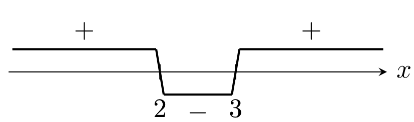
Examples

Example 5. Solve $x^2 - 5x + 6 > 0$.

Solution. Factor: $(x - 2)(x - 3) > 0$.

Roots: 2 and 3. Start on the far left: for very negative x, both factors are negative, so their product is positive. Flip signs at each root:

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of product	$+$	$-$	$+$

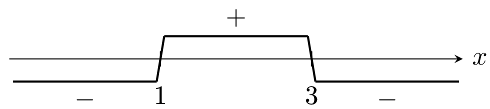


We need > 0, so the solution is $(-\infty, 2) \cup (3, \infty)$.

Example 6. Solve $-2x^2 + 8x - 6 \leq 0$.

Solution. Factor out -2: $-2(x^2 - 4x + 3) = -2(x-1)(x-3) \leq 0$.
Roots: 1 and 3. Start on the far left: x very negative gives $(x-1)(x-3) > 0$ but multiplied by -2 gives a negative value. Sign chart:

Interval	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
Sign of expression	$-$	$+$	$-$



We need ≤ 0, so: $(-\infty, 1] \cup [3, \infty)$.

Example 7 (No Real Roots)**.** Solve $x^2 + 4x + 5 < 0$.

Solution. Discriminant: $\Delta = 16 - 20 = -4 < 0$. No roots means the sign never changes. Check at $x = 0$: $x^2 + 4x + 5 = 5 > 0$.
Thus the quadratic is positive for all x, so $x^2 + 4x + 5 < 0$ has **no solutions**.

Quick Check

Solve the following quadratic inequalities:

4. $(x-3)(x+1) > 0$
5. $x^2 - 4x + 3 \leq 0$
6. $x^2 + 1 < 0$

Rational Inequalities

With fractions, you cannot multiply both sides by the denominator (because it could be positive or negative). Instead, analyze signs of numerator and denominator separately.

A fraction is:

- **positive** if numerator and denominator have the same sign,
- **negative** if they have opposite signs.

Example 8. Solve $\dfrac{x-4}{2x-10} \leq 0$.

Solution. We need the numerator and denominator to have opposite signs (or numerator = 0).
Case 1: $x - 4 \geq 0$ and $2x - 10 < 0$ gives $4 \leq x < 5$.
Case 2: $x - 4 \leq 0$ and $2x - 10 > 0$ — impossible.
Solution: $[4, 5)$.

Example 9. Solve $\dfrac{2x-1}{x+3} \geq 3$.

Solution. Move everything to one side:

$$\frac{2x-1}{x+3} - 3 \geq 0 \quad \Rightarrow \quad \frac{2x-1-3(x+3)}{x+3} = \frac{-x-10}{x+3} \geq 0.$$

A fraction is non-negative when numerator and denominator have the same sign.
Case 1: $-x-10 \geq 0$ and $x+3>0$ — impossible.
Case 2: $-x-10 \leq 0$ and $x+3<0$ gives $x \geq -10$ and $x<-3$.
Solution: $[-10,-3)$.

Inequalities With a Parameter

In many real problems, the inequality does not involve only x. It may also contain another quantity — a **parameter** — usually written as a, k, m, or c.

A parameter is a number whose value is *not specified in advance*. Depending on its value, the inequality may behave very differently.

For example, in the inequality

$$ax < 5,$$

we cannot simply divide by a, because we do not know whether a is positive, negative, or zero. And the sign of a determines whether the inequality flips or not.

When an inequality contains a parameter, you must consider separate cases:

$$a > 0, \qquad a = 0, \qquad a < 0.$$

This is necessary because dividing or multiplying by a depends on its sign.

Why do we need parameter inequalities?

Situations involving parameters appear naturally in:

- **Physics:** formulas involving mass, density, or acceleration a;
- **Engineering:** safety limits depending on material strength k;
- **Economics:** profit conditions depending on tax rate t;
- **Algebra:** studying families of functions $f_a(x)$ as a varies.

Parameters allow us to solve not just one problem, but an *entire family* of problems at once.

Real-Life Example: Safe Braking on Different Road Conditions

When a car brakes, the shortest stopping distance can be approximated by

$$d = \frac{v^2}{2\mu g},$$

where

- v is the car's speed,
- g is gravitational acceleration (approx. 9.8 m/s^2),
- μ is the friction coefficient between the tires and the road.

Different road conditions give different values of μ:

$$\mu = \begin{cases} 0.75 & \text{dry asphalt} \\ 0.40 & \text{wet road} \\ 0.20 & \text{icy road} \end{cases}$$

Suppose a driver sees an obstacle 25 meters ahead. We want all speeds v that are safe:

$$d \leq 25.$$

Using the formula:

$$\frac{v^2}{2\mu g} \leq 25.$$

Solve for v depending on μ:

$$v^2 \leq 50\mu g, \qquad v \leq \sqrt{50\mu g}.$$

Now evaluate for each road condition:

- **Dry road** ($\mu = 0.75$):
$$v \leq \sqrt{50 \cdot 0.75 \cdot 9.8} \approx 19.2 \text{ m/s} \approx 43 \text{ mph}.$$
- **Wet road** ($\mu = 0.40$):
$$v \leq \sqrt{50 \cdot 0.40 \cdot 9.8} \approx 14.0 \text{ m/s} \approx 31 \text{ mph}.$$
- **Icy road** ($\mu = 0.20$):
$$v \leq \sqrt{50 \cdot 0.20 \cdot 9.8} \approx 9.9 \text{ m/s} \approx 22 \text{ mph}.$$

Interpretation: The "safe speed" depends dramatically on the parameter μ. On dry pavement you can brake safely even at moderate speed, but on ice the safe speed drops to about 22 mph.
This example shows why inequalities with parameters are essential: the *same* inequality has different solutions depending on road conditions.

Examples

Example 10. Solve $ax - 5 < 0$ depending on the parameter a.

Solution. We have $ax < 5$.
Case 1: $a = 0$. The inequality becomes $-5 < 0$, which is always true. Solution: all real x.
Case 2: $a > 0$. Divide by a: $x < \frac{5}{a}$.
Case 3: $a < 0$. Dividing flips the inequality: $x > \frac{5}{a}$.

Example 11. Solve $x^2 > a$ depending on the parameter a.

Solution.
Case 1: $a < 0$. Since $x^2 \geq 0$ for all x, the inequality is always true. Solution: $(-\infty, \infty)$.
Case 2: $a = 0$. We need $x^2 > 0$, so $x \neq 0$. Solution: $(-\infty, 0) \cup (0, \infty)$.
Case 3: $a > 0$. We need $x^2 > a$, so $x < -\sqrt{a}$ or $x > \sqrt{a}$. Solution: $(-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$.

Inequalities in Real Life

Economics Example: Finding the Profitable Price Range

A company sells a product for price p dollars. Market research shows that the number of units sold is approximately

$$q(p) = 500 - 20p.$$

(500 is the maximum number of units the company can produce. As the price goes up, fewer people buy the product.)
The cost of producing q units is

$$C = 2000 + 4q,$$

where 2000 is the fixed cost, and 4 is cost per product, and the revenue is

$$R = pq. \qquad \text{(price times the number of products sold)}$$

The company wants to know for which prices the business is *profitable*:

$$R - C > 0.$$

Substitute $q = 500 - 20p$:

$$R = p(500 - 20p) = 500p - 20p^2.$$

The cost becomes:

$$C = 2000 + 4(500 - 20p) = 2000 + 2000 - 80p = 4000 - 80p.$$

Thus profit is:

$$\Pi(p) = R - C = (500p - 20p^2) - (4000 - 80p).$$

Simplify:

$$\Pi(p) = -20p^2 + 580p - 4000.$$

We want:

$$-20p^2 + 580p - 4000 > 0.$$

Divide by -20 and flip the sign:

$$p^2 - 29p + 200 < 0.$$

Factor:

$$(p - 8)(p - 25) < 0.$$

A product is negative when the factors have opposite signs:

$$8 < p < 25.$$

Interpretation: The company is profitable only if the price is between \$8 and \$25. Below \$8 they lose money because revenue is too small; above \$25, demand collapses and they don't sell enough units.

Key Takeaways

- **Linear inequalities**: Solve like equations, but flip the sign when multiplying/dividing by a negative.
- **Product inequalities**: A product is positive when factors have the same sign, negative when opposite signs.
- **Quadratic inequalities**: Factor, find roots, then determine signs on each interval.
- **Rational inequalities**: Analyze signs of numerator and denominator separately; never multiply by the denominator.
- **Interval notation**: Use (or) for excluded endpoints, [or] for included endpoints.
- **Parameter inequalities**: Consider cases $a > 0$, $a = 0$, $a < 0$ separately.

Quick Reference

Type	Method
Linear: $ax + b < c$	Isolate x; flip sign if dividing by negative
Product: $(x-a)(x-b) > 0$	Same signs $\Rightarrow$ positive; opposite $\Rightarrow$ negative
Quadratic: $ax^2 + bx + c > 0$	Factor, find roots, test intervals (snake method)
Rational: $\frac{p(x)}{q(x)} > 0$	Analyze signs separately; exclude $q(x) = 0$
With parameter	Consider cases: $a > 0$, $a = 0$, $a < 0$

Common Mistakes

- **Forgetting to flip the sign** when multiplying or dividing by a negative number.
- **Multiplying by a denominator** whose sign is unknown — this can reverse the inequality!
- **Misplacing endpoints**: Use [or] when the endpoint is included ($\leq$, $\geq$), and (or) when excluded ($<$, $>$).
- **Assuming $\frac{A}{B} > 1$ means $A > B$** — this only works if $B > 0$. Signs matter!
- **Forgetting to exclude points** where the denominator is zero in rational inequalities.

Classwork

1. Solve the linear inequalities:
   (a) $5x - 3 > 2x + 9$
   (b) $7 - 4x \leq 3$
2. Solve the quadratic inequalities:
   (a) $(x - 2)(x + 3) < 0$
   (b) $x^2 - 9 \geq 0$
3. Solve the rational inequality: $\dfrac{6}{x} < 2$
4. Solve the rational inequality: $\dfrac{x+2}{x-1} > 0$
5. Solve depending on the parameter a: $2x + a < 5$

Classwork Solutions

1. Linear inequalities:
   (a) $5x - 3 > 2x + 9 \Rightarrow 3x > 12 \Rightarrow x > 4$. Solution: $(4, \infty)$
   (b) $7 - 4x \leq 3 \Rightarrow -4x \leq -4 \Rightarrow x \geq 1$. Solution: $[1, \infty)$
2. Quadratic inequalities:
   (a) $(x-2)(x+3) < 0$: Roots at $x = 2$ and $x = -3$. Product is negative between the roots. Solution: $(-3, 2)$
   (b) $x^2 - 9 \geq 0 \Rightarrow (x-3)(x+3) \geq 0$: Non-negative outside the roots. Solution: $(-\infty, -3] \cup [3, \infty)$
3. $\dfrac{6}{x} < 2$: Cannot multiply by x without knowing its sign.

   Case 1: $x > 0$**.** Multiply by x: $6 < 2x \Rightarrow x > 3$. Combined with $x > 0$: $x > 3$.

   Case 2: $x < 0$**.** Multiply by x (flip!): $6 > 2x \Rightarrow x < 3$. Combined with $x < 0$: $x < 0$.

   Solution: $(-\infty, 0) \cup (3, \infty)$
4. $\dfrac{x+2}{x-1} > 0$: Positive when both factors have the same sign.
   - Both positive: $x > 1$
   - Both negative: $x < -2$

   Solution: $(-\infty, -2) \cup (1, \infty)$
5. $2x + a < 5 \Rightarrow 2x < 5 - a \Rightarrow x < \frac{5-a}{2}$. Solution: $\left(-\infty, \frac{5-a}{2}\right)$ for all values of a.

Homework

1. Solve the following linear inequalities:
   (a) $4x + 6 < 2x + 14$ (b) $-2(x + 3) < 10$ (c) $-2(x + 2) > 4 - x$
2. Solve the following inequalities:
   (a) $x^2 \leq 81$ (b) $x^2 \geq 100$ (c) $\sqrt{x} \leq 7$ (d) $\sqrt{x + 10} < 2$
3. Solve the following inequalities:
   (a) $(x - 3)(x + 7) > 0$ (b) $(2x + 1)(x - 5) \leq 0$ (c) $(x - 4)(3 - 2x) < 0$
4. Solve the quadratic inequalities:
   (a) $x^2 - 7x + 10 < 0$ (b) $2x^2 - 11x + 5 \geq 0$ (c) $3x^2 + x - 10 > 0$ (d) $x^2 + x - 20 \leq 0$ (e) $(x + 5)^2 \leq 0$ (f) $3(x - 2)^2 \geq 0$
5. Solve the inequalities:
   (a) $-x^2 + 4x - 3 > 0$ (b) $-2x^2 - 3x + 5 \leq 0$
6. Solve the rational inequalities:
   (a) $\frac{4}{x} > 1$ (b) $\frac{3}{x - 1} \leq 1$
7. M Solve the following rational inequalities:
   (a) $\frac{1 + 3x}{x - 2} > 2$ (b) $\frac{x + 1}{x - 5} \leq 0$ (c) $\frac{3x + 1}{x + 4} \geq 1$
8. M Find all values of a for which the following equation has a positive solution x.

   $$4 - a = \frac{2}{x - 1}$$

   (Hint: solve for x in terms of a and then determine for which a the result is positive.)
9. M Depending on the value of a, solve the inequality

   $$(a - 2)x > a^2 - 4.$$

10. H Find all values of a for which the inequality

    $$x^2 - ax + 4 > 0$$

    has no solutions, exactly one solution, or infinitely many solutions.
11. M Depending on the value of a, solve the inequality

    $$\sqrt{x} \geq a.$$

12. M Solve the following quadratic equations and inequalities:
    (a) $x^2 + 2x - 3 = 0$, $x^2 + 2x - 3 > 0$, $x^2 + 2x - 3 \leq 0$
    (b) $x^2 + 2x + 3 = 0$, $x^2 + 2x + 3 \geq 0$, $x^2 + 2x + 3 < 0$
    (c) $-x^2 + 6x - 9 = 0$, $-x^2 + 6x - 9 \geq 0$, $-x^2 + 6x - 9 < 0$
    (d) $3x^2 + x - 1 = 0$, $3x^2 + x - 1 \geq 0$, $3x^2 + x - 1 \leq 0$

Quick Check Answers

Linear Inequalities:

1. $3x - 7 < 5 \Rightarrow 3x < 12 \Rightarrow x < 4$. Solution: $(-\infty, 4)$
2. $4 - 2x \geq 10 \Rightarrow -2x \geq 6 \Rightarrow x \leq -3$. Solution: $(-\infty, -3]$
3. $2(x + 1) > 3x - 4 \Rightarrow 2x + 2 > 3x - 4 \Rightarrow 6 > x$. Solution: $(-\infty, 6)$

Quadratic Inequalities:

4. $(x - 3)(x + 1) > 0$: Positive outside roots. Solution: $(-\infty, -1) \cup (3, \infty)$
5. $x^2 - 4x + 3 \leq 0 \Rightarrow (x - 1)(x - 3) \leq 0$. Solution: $[1, 3]$
6. $x^2 + 1 < 0$: No solutions (sum of non-negative terms can't be negative)