MATH 7: HANDOUT 14

ABSOLUTE VALUE AND EQUATIONS WITH ABSOLUTE VALUE

Equations with Absolute Value

What is Absolute Value?

The **absolute value** of a number, written as $|x|$, means its *distance from zero* on the number line. Because distance is never negative:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

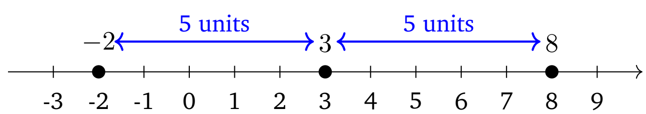
Example 1. $|5| = 5$ and $|-5| = 5$. Both are 5 units away from zero.

Geometric Interpretation: Distance Between Points

More generally, $|x - a|$ represents the **distance between** x **and** a on the number line.

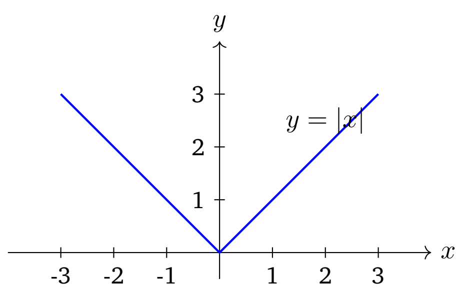
- $|x - 3|$ = distance from x to 3
- $|x + 2| = |x - (-2)|$ = distance from x to -2

This interpretation is powerful: the equation $|x - 3| = 5$ asks "which points are exactly 5 units away from 3?" The answer: $x = 8$ (to the right) and $x = -2$ (to the left).



Graph of Absolute Value

The graph of $y = |x|$ has a characteristic **V-shape** with a corner at the origin:



The "corner" occurs where the expression inside the absolute value equals zero. When solving equations with multiple absolute values, each one contributes a potential corner—these are the **critical points**.

Key Properties of Absolute Value

- $|a| = |-a|$ (absolute value ignores the sign)
- $|ab| = |a| \cdot |b|$ (absolute value of a product)
- $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ (absolute value of a quotient, $b \neq 0$)
- $|a + b| \leq |a| + |b|$ (**Triangle Inequality**)

How to Solve Absolute Value Equations

When you see $|f(x)| = k$, think: the inside of the bars can be either k or $-k$.

Solving Absolute Value Equations

1. **Isolate** the absolute value expression.
2. **Set up two cases:**
$$f(x) = k \quad \text{and} \quad f(x) = -k$$
(only makes sense when $k \geq 0$).
3. **Solve** each equation.
4. **Check your answers** in the original equation — sometimes extra roots sneak in!

Quick Check

1. What is $|-7|$? What is $|7|$?
2. Solve $|x| = 4$.
3. If $|x - 2| = 5$, what are the two possible values of x?

Critical Points and Intervals

When there are several absolute values (like $|x-3|+|2x+1|$), each expression inside can change sign at some point. Those x-values are called the **critical points** — where something "flips" from positive to negative.

- Find where each inside expression equals 0.
- Those points divide the number line into intervals.
- On each interval, you can *drop the bars* and simplify the signs consistently.

Domain Check (Why and When We Use It)

When solving an equation of the form
$$|f(x)| = g(x),$$
the expression on the left is always nonnegative. That means the right-hand side $g(x)$ must also be nonnegative for any real solution.

$$\text{If } g(x) < 0, \text{ the equation has no solution at that } x.$$

However, this condition is not an extra case you must solve separately. When you set up the two cases $f(x) = g(x)$ and $f(x) = -g(x)$, any x that makes $g(x) < 0$ will automatically be rejected by those case conditions.

So why check the domain first? Because it's *efficient*. It tells you right away which regions of x can possibly contain solutions and which can't. That saves time by letting you skip unnecessary cases.

Example 2 (Efficient Domain Check)**.** Solve $|2x - 5| = x - 7$.

Domain: $x - 7 \geq 0 \Rightarrow x \geq 7$. Since $x \geq 7$ implies $2x - 5 \geq 0$, we drop the bars: $2x - 5 = x - 7 \Rightarrow x = -2$. This doesn't satisfy $x \geq 7$, so there is **no solution.**

In summary:

- The **domain check is not strictly required** for correctness—case analysis will reject invalid values automatically.
- It is an **efficient shortcut**: it tells you immediately where solutions *might* exist.

Examples

Example 3 (The Simplest One)**.** Solve $|x| = 5$. We get $x = 5$ or $x = -5$. **Answer:** $x = \pm 5$

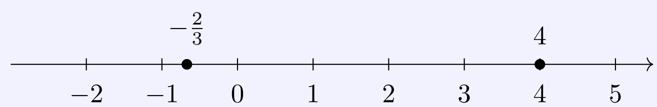
Example 4 (Shifted Inside)**.** Solve $|x - 3| = 7$. Either $x - 3 = 7 \Rightarrow x = 10$, or $x - 3 = -7 \Rightarrow x = -4$. **Answer:** $x = 10$ or $x = -4$

Simplifying a Sum of Absolute Values

Example 5. Simplify $|x - 4| + |3x + 2|$.
Step 1 (Find critical points). Each absolute value changes sign where its inside equals zero:

$$x - 4 = 0 \Rightarrow x = 4, \quad \text{and} \quad 3x + 2 = 0 \Rightarrow x = -\tfrac{2}{3}.$$

These two points split the number line into three regions.



Step 2 (Simplify in each region).

- **Region I:** $x \geq 4$. Both expressions are nonnegative:

$$|x - 4| = x - 4, \quad |3x + 2| = 3x + 2.$$

  Then $|x - 4| + |3x + 2| = (x - 4) + (3x + 2) = 4x - 2$.
- **Region II:** $-\frac{2}{3} \leq x < 4$. Here $x - 4 < 0$ but $3x + 2 \geq 0$:

$$|x - 4| = -(x - 4) = -x + 4, \quad |3x + 2| = 3x + 2.$$

  Then $|x - 4| + |3x + 2| = (-x + 4) + (3x + 2) = 2x + 6$.
- **Region III:** $x < -\frac{2}{3}$. Both expressions are negative:

$$|x - 4| = -(x - 4) = -x + 4, \quad |3x + 2| = -(3x + 2) = -3x - 2.$$

  Then $|x - 4| + |3x + 2| = (-x + 4) + (-3x - 2) = -4x + 2$.

Step 3 (Combine the results).

$$|x - 4| + |3x + 2| = \begin{cases} 4x - 2, & x \geq 4, \\ 2x + 6, & -\frac{2}{3} \leq x < 4, \\ -4x + 2, & x < -\frac{2}{3}. \end{cases}$$

This form shows exactly how the expression changes slope across the two "corners."

Absolute Value Equals a Linear Expression

Example 6. Solve $|x-4| = 2x-6$
Step 0 (Domain). Since the left side is an absolute value (always ≥ 0), we need the right side to be nonnegative:

$$2x-6 \geq 0 \;\Rightarrow\; x \geq 3.$$

So any solution must satisfy $x \geq 3$.

Critical point: The expression inside the absolute value changes sign at $x = 4$. This splits the number line into intervals to test: $[3, 4)$ and $[4, \infty)$.

Region I: $x \geq 4$. Then $|x-4| = x-4$, so

$$x-4 = 2x-6 \;\Rightarrow\; x = 2.$$

But $x = 2$ does *not* satisfy $x \geq 4$ (nor the domain $x \geq 3$), so $\boxed{\text{no solution from Case 1.}}$
Region II: $3 \leq x < 4$. Then $|x-4| = -(x-4) = -x+4$, so

$$-x+4 = 2x-6 \;\Rightarrow\; 3x = 10 \;\Rightarrow\; x = \frac{10}{3}.$$

Check the interval and domain: $\frac{10}{3} \approx 3.33$ is in $[3, 4)$, and

$$2x-6 = 2 \cdot \frac{10}{3} - 6 = \frac{20}{3} - \frac{18}{3} = \frac{2}{3} \geq 0,$$

so it satisfies the original equation.

Conclusion. The only solution is

$$\boxed{x = \frac{10}{3}}.$$

Absolute Value Equation with No Solution

Example 7. Solve $|x+2| = x-4$.
Step 0 (Domain). The left side is an absolute value (always ≥ 0), so we must have

$$x-4 \geq 0 \;\Rightarrow\; x \geq 4.$$

If $x < 4$, then the right side is negative and the equation is impossible.
Step 1 (Drop the bars using the domain). For all $x \geq 4$ we have $x+2 \geq 0$, so $|x+2| = x+2$. The equation becomes

$$x+2 = x-4,$$

which is impossible ($2 = -4$).
Conclusion. No x satisfies the equation.

$$\boxed{\text{No solution.}}$$

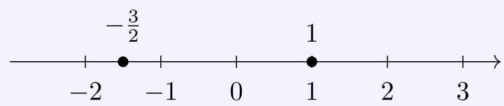
Two Absolute Values in an Equation

Example 8. Solve $|x-1| + |2x+3| = 5$.

Step 1 (Critical points). Each absolute value changes sign when its inside is zero:

$$x - 1 = 0 \Rightarrow x = 1, \quad \text{and} \quad 2x + 3 = 0 \Rightarrow x = -\tfrac{3}{2}.$$

These split the number line into three regions.



Step 2 (Solve by regions).

- **Region I:** $x \geq 1$. Both expressions are nonnegative:

  $$(x-1) + (2x+3) = 5 \Rightarrow 3x + 2 = 5 \Rightarrow x = 1.$$

  Valid, since $x = 1 \geq 1$.

- **Region II:** $-\frac{3}{2} \leq x < 1$. Here $x - 1 < 0$, $2x + 3 \geq 0$:

  $$-(x-1) + (2x+3) = 5 \Rightarrow -x + 1 + 2x + 3 = 5 \Rightarrow x + 4 = 5 \Rightarrow x = 1.$$

  But $x = 1$ is the right endpoint, not included (< 1), so exclude it.

- **Region III:** $x < -\frac{3}{2}$. Both expressions negative:

  $$-(x-1) - (2x+3) = 5 \Rightarrow -x + 1 - 2x - 3 = 5 \Rightarrow -3x - 2 = 5 \Rightarrow x = -\tfrac{7}{3}.$$

  This value ($-\frac{7}{3} \approx -2.33$) fits the region.

Step 3 (Verify).

$$x = 1 : |1-1| + |2(1)+3| = 0 + 5 = 5 \checkmark$$

$$x = -\tfrac{7}{3} : \left|-\tfrac{7}{3} - 1\right| + \left|2\left(-\tfrac{7}{3}\right) + 3\right| = \left|-\tfrac{10}{3}\right| + \left|-\tfrac{5}{3}\right| = \tfrac{10}{3} + \tfrac{5}{3} = 5 \checkmark$$

Conclusion.

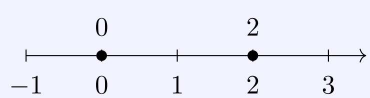
$$\boxed{x = 1 \quad \text{or} \quad x = -\tfrac{7}{3}.}$$

Quadratic Inside Absolute Value

Example 9. Solve $|x^2 - 4| = 2x$.
Note: This problem combines absolute values with quadratic equations from Handout 12.
Step 0 (Domain). Since the left side is ≥ 0, we need $2x \geq 0$, i.e.

$$x \geq 0.$$

So we only consider $x \geq 0$.
Step 1 (Critical points). The inside changes sign at $x^2 - 4 = 0 \Rightarrow x = \pm 2$. With the domain $x \geq 0$, we split into two intervals: $0 \leq x < 2$ and $x \geq 2$.



Region I: $x \geq 2$. Then $x^2 - 4 \geq 0$, so $|x^2 - 4| = x^2 - 4$.

$$x^2 - 4 = 2x \;\Rightarrow\; x^2 - 2x - 4 = 0 \;\Rightarrow\; x = 1 \pm \sqrt{5}.$$

Numerically, $1+\sqrt{5} \approx 3.236$ and $1-\sqrt{5} \approx -1.236$. Only $x = 1+\sqrt{5}$ lies in $x \geq 2$.
Region II: $0 \leq x < 2$. Then $x^2 - 4 < 0$, so $|x^2 - 4| = -(x^2 - 4) = -x^2 + 4$.

$$-x^2 + 4 = 2x \;\Rightarrow\; x^2 + 2x - 4 = 0 \;\Rightarrow\; x = -1 \pm \sqrt{5}.$$

Numerically, $-1+\sqrt{5} \approx 1.236$ and $-1-\sqrt{5} \approx -3.236$. Only $x = -1+\sqrt{5}$ lies in $[0, 2)$.
Step 2 (Check in the original equation).

$$x = 1+\sqrt{5}: \quad |(1+\sqrt{5})^2 - 4| = |2+2\sqrt{5}| = 2+2\sqrt{5}, \quad 2x = 2(1+\sqrt{5}) = 2+2\sqrt{5} \quad \checkmark$$

$$x = -1+\sqrt{5}: \quad |(-1+\sqrt{5})^2 - 4| = |2-2\sqrt{5}| = 2\sqrt{5}-2, \quad 2x = 2(-1+\sqrt{5}) = -2+2\sqrt{5} \quad \checkmark$$

Conclusion. Both candidates work and satisfy $x \geq 0$:

$$\boxed{x = 1+\sqrt{5} \quad \text{or} \quad x = -1+\sqrt{5}.}$$

Quick Check

4. What are the critical points for $|x-2| + |x+3|$?
5. Solve $|x-1| = 2x+1$. (Hint: check the domain first!)
6. True or False: $|a+b| = |a| + |b|$ for all real numbers a and b.

Common Mistakes to Avoid

- **Forgetting that $|x| = k$ has no solution when $k < 0$.** Absolute values are always ≥ 0, so $|x| = -3$ is impossible.
- **Not checking solutions.** When you split into cases, always verify that each solution actually satisfies the original equation *and* falls within the correct interval.
- **Thinking $|a+b| = |a|+|b|$.** This is false in general! For example, $|3+(-5)| = 2$, but $|3|+|-5| = 8$.
- **Forgetting that $|a| = |-a|$.** This identity is useful for simplifying expressions like $|x-5|+|5-x| = 2|x-5|$.
- **Missing critical points.** When multiple absolute values are present, find *all* points where an expression inside changes sign.

Classwork

Warm-up

1. $|x| = 6$
2. $|x + 2| = 8$
3. $|2x| = 10$
4. Simplify $|x - 3| + |2x + 5|$

Level Up

1. $|x - 5| = 2x$
2. $|3x - 1| = 5$
3. $|x + 2| + 3 = 7$
4. $|x| = |x - 4|$

Advanced

1. $|2x - 1| = |x + 3|$
2. $|x + 3| - |x - 2| = 1$
3. $|3x - 7| = 2|x + 1|$
4. $|x^2 - 4| = 4x - 4$

Challenge

1. $|x^2 - 5x + 6| = 4$
2. $|x + 1| + |2x - 3| = 7$
3. $|x^2 - 4x + 3| + |x - 1| = 6$

Classwork Solutions

Warm-up

1. $x = \pm 6$
2. $x = 6$ or $x = -10$
3. $x = \pm 5$
4. Critical points: $x = -\frac{5}{2}$ and $x = 3$.
$$|x-3| + |2x+5| = \begin{cases} 3x+2, & x \geq 3 \\ -x+8, & -\frac{5}{2} \leq x < 3 \\ -3x-2, & x < -\frac{5}{2} \end{cases}$$

Level Up

1. Domain: $x \geq 0$. Cases give $x = \frac{5}{3}$ (valid) and $x = -5$ (invalid). Answer: $x = \frac{5}{3}$
2. $x = 2$ or $x = -\frac{4}{3}$
3. $|x+2| = 4 \Rightarrow x = 2$ or $x = -6$
4. $x - 4 = x$ or $x - 4 = -x \Rightarrow x = 2$

Advanced

1. $2x - 1 = x + 3$ or $2x - 1 = -(x+3) \Rightarrow x = 4$ or $x = -\frac{2}{3}$
2. Critical points at $x = -3$ and $x = 2$. Answer: $x = 0$
3. $3x - 7 = 2(x+1)$ or $3x - 7 = -2(x+1) \Rightarrow x = 9$ or $x = 1$
4. Domain: $x \geq 1$. Answer: $x = 4$ or $x = -2 + 2\sqrt{3}$

Challenge

1. Factor inside: $x^2 - 5x + 6 = (x-2)(x-3)$. Answer: $x = \frac{5 \pm \sqrt{17}}{2}$
2. Critical points at $x = -1$ and $x = \frac{3}{2}$. Answer: $x = 3$ or $x = -\frac{5}{3}$
3. Factor: $(x-1)(x-3)$. Critical points at $x = 1$ and $x = 3$. Answer: $x = 4$ or $x = \frac{5-\sqrt{33}}{2}$

Homework

1. **Evaluate numerically.**

   (a) If $x = -3$, calculate $|x| + |x+2|$.

   (b) If $x = 2$ and $y = -4$, find $|x-y| + |y-x|$.

2. M **Simplify piecewise.** Write each expression as a piecewise function (similar to Example 3 in class). That is, find how the expression changes in different intervals of x.

   (a) $|2x+3| + |x-1|$

   (b) $|x+6| + |4x-2|$

   (c) *Bonus question:* Sketch the graph of one of them. How many "corners" does it have?

3. **Solve.** Show your reasoning: isolate the absolute value, set up cases, and check solutions.

   (a) $|x+5| = 7$

   (b) $|3x-2| = 4$

   (c) $|x-1| + 2 = 6$

4. M **Solve.**

   (a) $|x+3| = |x-1|$

   (b) $|x+2| + |x-1| = 6$

   (c) $|x-3| + |x+1| = 10$

   (d) $|x-2025| + |2025-x| = 2026$
   Hint: Notice that $|a| = |-a|$ for any a.

5. M Solve the equation $|x-3| + |x+4| = 7$.

   (*Hint:* If $A(3)$ and $B(-4)$ are two points on a number line, the distance between them is $|A-B| = |3-(-4)| = 7$. Try extending this idea to the equation $|x-3| + |x+4| = 7$. Where is x located if the total distance to A and B is always 7?)

6. H Solve $|x^2 - 4x + 3| = 5$.

7. H Solve the equation:

   $$|x^3 - 4x| = 3x.$$

   (*Hint:* First identify where each side is nonnegative, then consider cases depending on the sign of $x^3 - 4x$.)

8. H For what values of a and b does the equation

   $$|x-a| + |x-b| = 10$$

   represent a *line segment of length 10* on the number line? Find a and b, and explain briefly what this means geometrically.

   (*Hint:* The sum of distances from x to a and b is constant — what does that mean? Find a and b if the segment's length is 10.)

Quick Check Answers

Quick Check 1:

1. $|-7| = 7$ and $|7| = 7$
2. $x = 4$ or $x = -4$
3. $x - 2 = 5 \Rightarrow x = 7$, or $x - 2 = -5 \Rightarrow x = -3$

Quick Check 2:

4. Critical points: $x = 2$ and $x = -3$
5. Domain: $2x + 1 \geq 0 \Rightarrow x \geq -\frac{1}{2}$. For $x \geq 1$: $x - 1 = 2x + 1 \Rightarrow x = -2$ (invalid). For $-\frac{1}{2} \leq x < 1$: $-(x - 1) = 2x + 1 \Rightarrow x = 0$. Answer: $x = 0$
6. False. Counterexample: $|3 + (-5)| = 2$, but $|3| + |-5| = 8$