MATH 7: HANDOUT 12

Quadratic Equations I

Quadratic Equations

Quadratic equations appear everywhere in mathematics and science. They describe the path of a thrown ball, the shape of a satellite dish, the design of bridges, and even profit calculations in business. Whenever something grows, curves, or bounces, a quadratic is hiding there!

Where do we see quadratic equations?

- **Physics:** The path of a thrown ball (a parabola).
- **Engineering:** Satellite dishes and car headlights use parabolic shapes to focus light or sound.
- **Architecture:** Arches and bridges often follow quadratic curves.
- **Economics:** Profit and cost models are often quadratic — they rise, then fall.
- **Games:** In Angry Birds or basketball, the bird/ball follows a parabolic trajectory.

Quadratic vs. Linear Equations

Before today, most of the equations we solved were **linear**:

$$ax + b = 0,$$

which always have exactly one solution, $x = -\frac{b}{a}$.

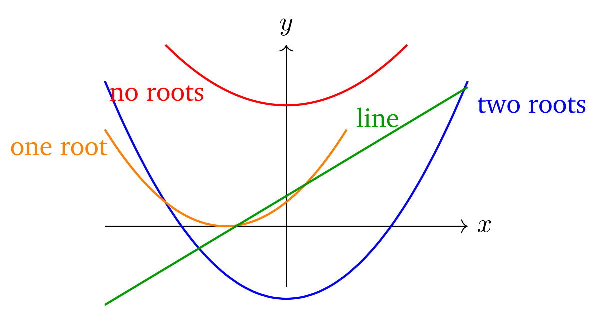
A **quadratic equation** is any equation of the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are numbers and $a \neq 0$. It can have *two*, *one*, or *no* real solutions depending on the discriminant $D = b^2 - 4ac$. The graph of a quadratic function,

$$y = ax^2 + bx + c,$$

is a smooth U-shaped curve called a **parabola**. We'll learn later how to draw parabolas and how their shapes change with different a, b, and c. For now, remember that this is the first big difference between straight lines and parabolas: a line crosses the x-axis once, but a parabola can cross it twice, just touch it, or miss it entirely.



Linear equations: at most 1 solution. **Quadratic equations:** up to 2 solutions. **Cubic (3rd-degree) equations:** up to 3 solutions. **Quartic (4th-degree) equations:** up to 4 solutions.

Note

Why "quadratic"? The word comes from Latin *quadratus*, meaning "square." A quadratic equation always involves the square of the unknown, x^2.

A Bit of History

Quadratic equations are ancient! The Babylonians knew how to solve some of them as early as 2000 BCE, using clay tablets and geometry. More than a thousand years later, the Persian mathematician **Al-Khwarizmi** (whose name gave us the word *algorithm*) explained systematic ways to solve quadratics — but without negative numbers or symbols! He did everything with words and diagrams. Centuries later, the formula we use today emerged in Europe once algebraic notation became common.

Completing the Square

Fun Fact

In 820 CE, Al-Khwarizmi solved quadratic equations by drawing geometric diagrams — he called the process "al-jabr," meaning "reunion of broken parts." That word gave us modern "algebra"!

One of the oldest and most elegant methods to solve a quadratic is called **completing the square**. Let's see how it works with an example.

Example 1. Solve $x^2 + 6x + 2 = 0$.

$$\begin{aligned} x^2 + 6x + 2 &= x^2 + 2 \cdot 3x + 9 - 7 \\ &= (x+3)^2 - 7. \end{aligned}$$

Thus,

$$(x+3)^2 = 7, \quad \Rightarrow \quad x + 3 = \pm\sqrt{7}.$$

$$\boxed{x = -3 \pm \sqrt{7}}.$$

This method always works — and it's actually how the famous quadratic formula was discovered.

The Idea in General (when $a = 1$)

Start with

$$x^2 + bx + c = 0.$$

Add and subtract $\left(\frac{b}{2}\right)^2$ to form a perfect square:

$$\begin{aligned} x^2 + bx + c &= x^2 + 2 \cdot \tfrac{b}{2}x + c \\ &= \left(x + \tfrac{b}{2}\right)^2 - \tfrac{b^2}{4} + c \\ &= \left(x + \tfrac{b}{2}\right)^2 - \tfrac{b^2 - 4c}{4}. \end{aligned}$$

Let $D = b^2 - 4c$ (the famous **discriminant**). Then

$$(x + \tfrac{b}{2})^2 = \tfrac{D}{4}.$$

Taking square roots:

$$x + \tfrac{b}{2} = \pm\tfrac{\sqrt{D}}{2}, \quad \text{so} \quad \boxed{x = \frac{-b \pm \sqrt{D}}{2}}.$$

When $a \neq 1$

Start with the general quadratic

$$ax^2 + bx + c = 0, \qquad a \neq 0.$$

Divide both sides by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Move the constant to the right:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $\left(\frac{b}{2a}\right)^2$ to both sides to complete the square:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

The left side is a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

Let $D = b^2 - 4ac$ (the **discriminant**). Then

$$\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}.$$

Take square roots (remember the $\pm$):

$$x + \frac{b}{2a} = \pm\frac{\sqrt{D}}{2a}.$$

Solve for x:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

Example 2 (With $a \neq 1$)**.** Solve $2x^2 + 7x + 3 = 0$ by completing the square.

Divide by 2:

$$x^2 + \frac{7}{2}x + \frac{3}{2} = 0.$$

Move the constant:

$$x^2 + \frac{7}{2}x = -\frac{3}{2}.$$

Complete the square by adding $\left(\frac{7}{4}\right)^2 = \frac{49}{16}$:

$$x^2 + \frac{7}{2}x + \frac{49}{16} = -\frac{3}{2} + \frac{49}{16} = -\frac{24}{16} + \frac{49}{16} = \frac{25}{16}.$$

The left side is a perfect square:

$$\left(x + \frac{7}{4}\right)^2 = \frac{25}{16}.$$

Take square roots:

$$x + \frac{7}{4} = \pm\frac{5}{4}.$$

So

$$x = -\frac{7}{4} + \frac{5}{4} = -\frac{1}{2} \quad \text{or} \quad x = -\frac{7}{4} - \frac{5}{4} = -3.$$

$$\boxed{x = -\frac{1}{2} \text{ or } x = -3}.$$

Summary

The Discriminant $D = b^2 - 4ac$**:**

- If $D > 0$, two distinct real roots (the parabola crosses the x-axis at two different points).
- If $D = 0$, one double root (the parabola just touches the x-axis at exactly one point).
- If $D < 0$, no real roots (the parabola does not touch the x-axis).

Common Mistakes to Avoid

- Don't forget the $\pm$ when taking square roots!
- Remember: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.
- Check that $a \neq 0$ — otherwise it's not a quadratic!
- When $D < 0$, there are no *real* solutions (but complex solutions exist — you'll learn about these later!)

More Practice

Example 3. Solve $2x^2 - 3x - 2 = 0$.

$$a = 2,\ b = -3,\ c = -2.$$
$$D = (-3)^2 - 4 \cdot 2 \cdot (-2) = 9 + 16 = 25.$$
$$x = \frac{-(-3) \pm \sqrt{25}}{2 \cdot 2} = \frac{3 \pm 5}{4}.$$
$$\Rightarrow x_1 = 2, \quad x_2 = -\tfrac{1}{2}.$$

Example 4. Solve $x^2 + 4x + 5 = 0$.

$$D = 4^2 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4 < 0.$$

No real solutions.

Example 5. Solve $x^2 - 6x + 9 = 0$.

$$D = (-6)^2 - 4 \cdot 1 \cdot 9 = 36 - 36 = 0.$$
$$x = \frac{6}{2} = 3.$$

Only one (double) root: $x = 3$. This means the parabola just kisses the x-axis at the point $(3, 0)$.

Factoring (when it works!)

Sometimes quadratics factor nicely, making them very easy to solve. The key idea: if a product equals zero, then at least one of the factors must be zero.

Why does this work? Remember that $(x-2)(x-3)$ means: first subtract 2 from x, then subtract 3 from x, and multiply the results. If the product is zero, then either $x - 2 = 0$ (so $x = 2$) or $x - 3 = 0$ (so $x = 3$).

Let's expand $(x-2)(x-3)$ to see the pattern:

$$\begin{aligned}(x-2)(x-3) &= x \cdot x + x \cdot (-3) + (-2) \cdot x + (-2) \cdot (-3) \\ &= x^2 - 3x - 2x + 6 \\ &= x^2 - 5x + 6.\end{aligned}$$

Notice: the coefficient of x is $-5 = (-2) + (-3)$, and the constant term is $6 = (-2) \times (-3)$.

The pattern: If we can write

$$x^2 + bx + c = (x - r)(x - s),$$

then expanding the right side gives:

$$x^2 - (r + s)x + rs.$$

Matching coefficients: $b = -(r + s)$ and $c = rs$.

So to factor $x^2 + bx + c$, we look for two numbers r and s such that:

- $r + s = -b$ (they add to the opposite of the coefficient of x), and
- $r \times s = c$ (they multiply to the constant term).

Example 6. Solve $x^2 - 5x + 6 = 0$.

We need two numbers that add to $-(-5) = 5$ and multiply to 6. Those numbers are 2 and 3 (since $2 + 3 = 5$ and $2 \times 3 = 6$).

So

$$x^2 - 5x + 6 = (x - 2)(x - 3) = 0.$$

Therefore $x - 2 = 0$ or $x - 3 = 0$, which gives

$$\boxed{x = 2 \text{ or } x = 3}.$$

Example 7. Solve $x^2 + 7x + 12 = 0$.

We need two numbers that add to -7 and multiply to 12. Since both the sum and product are positive, both numbers must be negative. Those are -3 and -4 (since $(-3) + (-4) = -7$ and $(-3) \times (-4) = 12$).

$$x^2 + 7x + 12 = (x - (-3))(x - (-4)) = (x + 3)(x + 4) = 0.$$

So $x = -3$ or $x = -4$.

When does factoring work? This method only works when the roots are "nice" whole numbers or simple fractions. For most quadratics (like $x^2 - 5x + 5 = 0$), we need completing the square or the quadratic formula.

In fact, a quadratic with integer coefficients factors nicely *only if* its discriminant is a perfect square!

Factored Form

General Factored Form: If x_1 and x_2 are the two roots of $ax^2 + bx + c = 0$, then the quadratic can be written as:

$$a(x - x_1)(x - x_2) = 0.$$

This always works, even when x_1 and x_2 involve square roots!
For example: the roots of $x^2 - 5x + 6 = 0$ are $x_1 = 2$ and $x_2 = 3$, so we can write

$$x^2 - 5x + 6 = 1 \cdot (x - 2)(x - 3).$$

Cubic and Quartic Equations

But if quadratics can have up to 2 solutions, what about higher-degree equations? What happens for higher powers of x?

For **cubic equations** ($ax^3 + bx^2 + cx + d = 0$), clever mathematicians in the 1500s finally discovered how to solve them.

The formulas do exist... but they are *terrifying*. For example, one case looks like this:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

where the cubic is written as $x^3 + px + q = 0$. (Don't worry — no one expects you to remember that!)

This formula was so complicated that even professional mathematicians at the time were astonished to discover that sometimes it worked only if you went through **imaginary numbers** along the way — a completely new idea in the 1500s!

For **quartic equations** ($ax^4 + bx^3 + cx^2 + dx + e = 0$), the Italian mathematician **Lodovico Ferrari** found a formula in 1540 — even more complicated than Cardano's cubic formula.

The Cardano-Tartaglia Feud

The story of how the cubic formula came to be published is one of the most dramatic episodes in the history of mathematics.

Niccolò Tartaglia (1499–1557) was a self-taught mathematician from Brescia, Italy. His nickname "Tartaglia" means "stammerer" — as a boy, he was slashed across the face by a French soldier during an invasion, leaving him with a severe speech impediment. Despite his humble origins and lack of formal education, he became one of the finest algebraists of his era.

In 1535, Tartaglia discovered how to solve certain types of cubic equations, a problem that had defeated mathematicians for centuries. He kept his method secret, as was common at the time — mathematical duels were public spectacles, and having a secret weapon meant winning prizes, prestige, and students.

Enter **Gerolamo Cardano** (1501–1576), a physician, astrologer, and mathematician from Milan. Cardano was desperate to learn Tartaglia's method. After much pleading, Tartaglia finally revealed his secret in 1539 — but only after Cardano swore a solemn oath never to publish it.

Cardano kept his word, technically. But then his student **Lodovico Ferrari** discovered how to solve quartic equations — and Ferrari's method *required* the cubic formula as a step. Meanwhile, Cardano learned that an earlier mathematician, **Scipione del Ferro**, had actually discovered the cubic formula before Tartaglia, but had also kept it secret until his death.

Feeling that the oath no longer applied (since del Ferro had priority), Cardano published everything in his 1545 masterpiece *Ars Magna* ("The Great Art"). He gave credit to both del Ferro and Tartaglia — but Tartaglia was furious.

What followed was a bitter, public feud lasting years. Tartaglia accused Cardano of theft and treachery. Cardano, who disliked public confrontation, sent Ferrari to defend his honor. In 1548, Tartaglia and Ferrari met in a public mathematical duel in Milan. Ferrari won decisively, and Tartaglia's reputation never recovered.

Today, the formula is usually called **Cardano's formula**, though some call it the Cardano-Tartaglia formula. The controversy reminds us that mathematics is a deeply human endeavor — full of ambition, jealousy, and broken promises, as well as brilliant ideas.

Equations of Higher Degrees

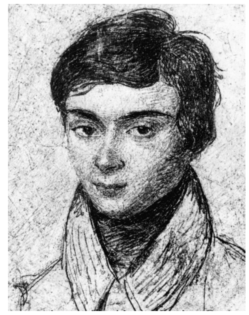
But for **degree 5 or higher**, something truly remarkable happens: there is *no general algebraic formula* using only arithmetic operations and radicals (square roots, cube roots, etc.)!

This stunning fact was first rigorously proved by the Norwegian mathematician **Niels Henrik Abel** in 1824. He showed that while formulas exist for degrees 2, 3, and 4, the pattern cannot continue to degree 5. The result is known as the **Abel-Ruffini theorem**.

A few years later, the young French genius **Évariste Galois** went even further. Before his tragic death in a duel at age 20 in 1832, Galois developed a complete theory (now called **Galois theory**) that explains *exactly which* polynomial equations can be solved by radicals and which cannot. His work revolutionized algebra and connected the solvability of equations to deep properties of symmetry groups.

Both Abel and Galois died tragically young — Abel at 26 from tuberculosis, Galois at 20 from a duel wound. Yet their discoveries remain among the most beautiful and profound results in all of mathematics.

N. H. Abel
(1802–1829)



É. Galois
(1811–1832)

Homework

1. Solve by any method:
   (a) $3x^2 - 12 = 0$
   (b) $(x-1)^2 = 6$
   (c) $(2x+1)^2 = 9$
   (d) $x^2 + 6x + 9 = 4$

2. Solve each of the following equations. Show all the steps of your reasoning clearly.
   (a) $x^2 - 6x + 5 = 0$
   (b) $x^2 - 5x + 5 = 0$
   (c) $x^2 = 1 + x$
   (d) $x^2 + 10x + 16 = 0$
   (e) $x^2 + 8x + 16 = 0$
   (f) $x^2 - 7x + 11 = 0$
   (g) $x^2 + 3x + 1 = 0$
   (h) $2x(3-x) = 1$
   (i) $\dfrac{x}{x-2} = x - 1$ (Hint: multiply both sides by $x-2$ first)
   (j) $x^3 + 4x^2 - 45x = 0$

3. Find all x such that
$$x^4 - 10x^2 + 9 = 0.$$

4. Solve for $x > 0$:
$$x - 3\sqrt{x} - 10 = 0.$$
   (Hint: Let $u = \sqrt{x}$.)

5. For what values of k does the equation
$$x^2 - (k+3)x + 4k = 0$$
   have exactly one real solution? (Hint: recall what $D = 0$ means.)

6. **Ancient Indian mathematics.** Indian mathematicians of the 9th century already knew methods for solving quadratic equations! Try this famous problem by the mathematician **Mahāvīra** (circa 850 CE):

   One-third of a herd of elephants and three times the square root of the remaining elephants were seen on a mountain slope; and in a lake were seen a male elephant along with three female elephants forming the final remainder. How many elephants were there in total?

7. Another puzzle, from the 12th-century Indian mathematician **Bhāskara II**:

   Out of a party of monkeys, the square of one-fifth of their number, diminished by three, went into a cave. The one remaining monkey was climbing up a tree. What was the total number of monkeys?

8. **Projectile motion.** A ball is thrown upward from the ground with an initial velocity of 20 m/s. Its height after t seconds is given by
$$h(t) = 20t - 5t^2.$$
   (a) When does the ball hit the ground?
   (b) When is the ball at a height of 15 meters? (There are two answers — why?)
   (c) What is the maximum height the ball reaches, and when does it occur?

9. **Exploring minimum and maximum values.**
   (a) Use formula $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{D}{4}$ to show that for any x,
$$x^2 + bx + c \geq -\frac{D}{4},$$
   with equality only when $x = -\frac{b}{2}$.

(b) Find the smallest possible value of the expression

$$x^2 + 4x + 2.$$

(c) Given a positive number a, find the largest possible value of

$$x(a - x).$$

(Your answer will depend on a.)

*10. **Nested radicals.** Simplify $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$ by assuming it equals some value x and deriving an equation for x.

*11. **Challenge: an iterative sequence.** Consider the sequence

$$x_1 = 1, \quad x_2 = \frac{x_1}{2} + \frac{1}{x_1}, \quad x_3 = \frac{x_2}{2} + \frac{1}{x_2}, \quad \ldots$$

Compute the first several terms. Does the sequence seem to be increasing, decreasing, or approaching some value? If it seems to settle, can you guess that value? (Hint: solve the equation $x = \frac{x}{2} + \frac{1}{x}$.)