MATH 7: HANDOUT 10

DISTRIBUTIONS. POKER PROBABILITIES

Probability Distributions and Histograms

A **probability distribution** describes all possible outcomes of an experiment together with their probabilities. It tells us how the total probability of 1 is distributed among the different outcomes.

Definition

A **probability distribution** lists all possible outcomes of an experiment together with the probability of each one.

If an experiment can have outcomes $x_1, x_2, \dots, x_n$, and the probability of each outcome is $p_1, p_2, \dots, p_n$, then the probability distribution can be written as a table:

Outcome	x_1	x_2	$\dots$	x_n
Probability	p_1	p_2	$\dots$	p_n

where

$$p_1 + p_2 + \cdots + p_n = 1.$$

This means that the total probability of all possible outcomes must always add up to 1 (something must happen!).

Rolling a Die

Let X be the number rolled on a fair six-sided die.

X	1	2	3	4	5	6
$P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Every outcome is equally likely, so the distribution is *uniform*.

Number of Heads in 3 Coin Tosses

Let X be the number of Heads when a fair coin is tossed 3 times.

$$P(X=0) = \frac{1}{8}, \quad P(X=1) = \frac{3}{8}, \quad P(X=2) = \frac{3}{8}, \quad P(X=3) = \frac{1}{8}.$$

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This distribution comes from the binomial formula with $n = 3$ and $p = \frac{1}{2}$.

Number of Heads in 4 Coin Tosses

Let X be the number of Heads when a fair coin is tossed 4 times. Then X can take values $0, 1, 2, 3, 4$, and by the binomial formula:

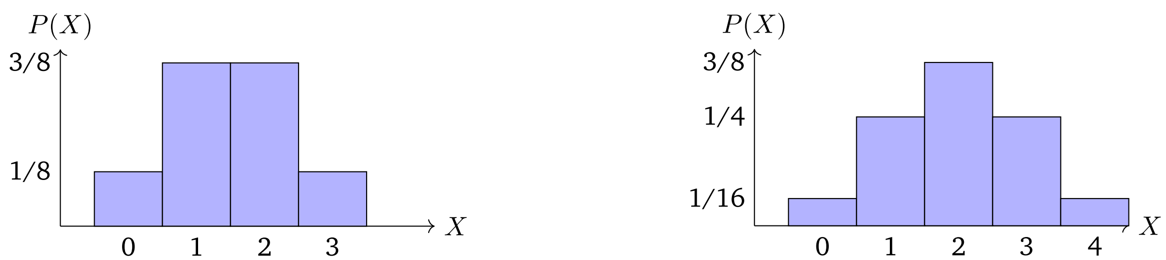
$$P(X = k) = \binom{4}{k}\left(\frac{1}{2}\right)^4.$$

X	0	1	2	3	4
$P(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The probabilities add up to 1, and the distribution is perfectly symmetric.

Visualizing with Histograms

A convenient way to show a probability distribution is with a **histogram** (or bar chart). The height of each bar represents the probability of that outcome.



3 Coin Tosses *4 Coin Tosses*

Comparison of probability distributions for 3 and 4 coin tosses.

- The total area (sum of bar heights) equals 1.
- The shape shows which outcomes are most likely.
- For symmetric distributions (like fair coin tosses), the graph looks the same on both sides.

Poker Probabilities

In five-card poker, a player is dealt 5 cards from a standard 52-card deck (4 suits: ♠, ♡, ♢, ♣, and 13 ranks: A, 2, 3, . . . , 10, J, Q, K).

Rule about Aces for straights: the Ace may be *low* ($A, 2, 3, 4, 5$) or *high* ($10, J, Q, K, A$), but it does *not* wrap around (e.g., $Q, K, A, 2, 3$ is not a straight).

Counting Toolkit

- **Sample space:** The total number of 5-card hands is

$$\binom{52}{5} = 2{,}598{,}960.$$

- **Combinations:** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ counts the ways to pick k objects from n without order.
- **Multiplication principle:** If choices split into independent stages, multiply the counts.
- **Careful exclusions:** When a class (e.g., straight) includes a special sub-class (e.g., straight flush), subtract those if you want the "non-flush" version.

Counting each hand type

We list *counts*; probabilities are counts divided by $\binom{52}{5}$.

Royal Flush $10, J, Q, K, A$ all in the same suit.(Example: 10♣, J♣, Q♣, K♣, A♣)
Exactly one per suit $\Rightarrow 4$ total.

Straight Flush Five consecutive ranks, all same suit, *excluding* royal flushes. (Example: 6♡, 7♡, 8♡, 9♡, 10♡)
There are 10 possible rank runs for a straight (A to 5, 2 to 6, . . . , 10 to A). With 4 suits, that gives $10 \cdot 4 = 40$ straight flushes *including* the 4 royals, so

$$\text{Straight flush (non-royal)} = 40 - 4 = 36.$$

Four of a Kind Four cards of the same rank plus one kicker. (Example: K♡, K♠, K♢, K♣, 2♣)
Choose the rank (13 ways) and the kicker (any of the remaining $52 - 4 = 48$ cards):

$$13 \cdot 48 = 624.$$

Full House Three of one rank and two of another. (Example: K♡, K♠, K♢, 4♠, 4♣)
Choose triple's rank (13) and suits $\binom{4}{3}$; choose pair's rank (12) and suits $\binom{4}{2}$:

$$13\binom{4}{3} \cdot 12\binom{4}{2} \;=\; 13 \cdot 4 \cdot 12 \cdot 6 \;=\; 3{,}744.$$

Flush Five cards in the same suit, not consecutive. (Example: 3♡, 6♡, 8♡, J♡, A♡)
Choose suit (4) and 5 ranks from 13: $4\binom{13}{5}$, then subtract all straight flushes (the 40 above):

$$4\binom{13}{5} - 40 \;=\; 5{,}148 - 40 \;=\; 5{,}108.$$

Straight Five consecutive ranks with mixed suits allowed (Ace low or high), but *not* all one suit. (Example: 5♡, 6♠, 7♢, 8♠, 9♣)
For each of 10 rank runs, each of the 5 cards can be any suit: $10 \cdot 4^5 = 10{,}240$. Subtract the 40 straight flushes:

$$10 \cdot 4^5 - 40 \;=\; 10{,}200.$$

Three of a Kind Three of one rank + two different kickers of distinct ranks. (Example: K♡, K♠, K♢, 4♠, 2♣)
Choose triple's rank (13) and suits $\binom{4}{3}$; choose two distinct kicker ranks $\binom{12}{2}$; choose suits of the two singletons 4^2:

$$13\binom{4}{3}\binom{12}{2}4^2 \;=\; 13\cdot 4\cdot 66\cdot 16 \;=\; 54{,}912.$$

Two Pairs Two different pairs + one kicker. (Example: K♡, K♠, 10♢, 10♠, 4♣)
Choose the two pair ranks $\binom{13}{2}$; choose suits for each pair $\binom{4}{2}^2$; choose kicker rank (11) and suit (4):

$$\binom{13}{2}\binom{4}{2}^2\cdot 11\cdot 4 \;=\; 78\cdot 36\cdot 44 \;=\; 123{,}552.$$

One Pair One pair + three kickers of distinct ranks. (Example: K♡, K♠, Q♢, 4♠, 2♣)
Choose pair rank (13) and suits $\binom{4}{2}$; choose three distinct kicker ranks $\binom{12}{3}$; choose suits of each singleton (4^3):

$$13\binom{4}{2}\binom{12}{3}4^3 \;=\; 13\cdot 6\cdot 220\cdot 64 \;=\; 1{,}098{,}240.$$

Nothing (High Card) No pair, not a straight, not a flush.
Subtract all other categories from $\binom{52}{5}$, or use the standard result:

$$1{,}302{,}540.$$

Common Pitfalls

- **Ace usage in straights:** Only $A2345$ and $10JQKA$ are valid edge-cases. No wrap-around like $QKA23$.
- **Avoid double counting:** If you count all flushes as $4\binom{13}{5}$, remember to subtract straight flushes.
- **Distinct ranks for kickers:** In *three of a kind* and *one pair*, the extra cards must be of different ranks (otherwise you create two pair or a full house).

Probabilities and Odds

Divide each count by $\binom{52}{5} = 2{,}598{,}960$.

Combination	Count	Probability	Odds (about)
Royal Flush	4	0.000154%	1 : 649,740
Straight Flush	36	0.00139%	1 : 72,192
Four of a Kind	624	0.0240%	1 : 4,165
Full House	3,744	0.1441%	1 : 693
Flush (non-straight)	5,108	0.1965%	1 : 508
Straight (non-flush)	10,200	0.3925%	1 : 254
Three of a Kind	54,912	2.1128%	1 : 46.3
Two Pairs	123,552	4.7539%	1 : 20.0
One Pair	1,098,240	42.2569%	1 : 1.37
Nothing (High Card)	1,302,540	50.1177%	≈ 1 : 2

Worked Idea

Why is the total $2{,}598{,}960$**?** Choosing any 5-card set from 52 without order is $\binom{52}{5}$. If you tried to count with order first $(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)$, you would then divide by $5!$ because the 5 cards can be ordered in $5!$ ways that represent the same hand.

Other Types of Poker and Sample Probabilities

While classical **five-card draw** is the simplest setting for combinatorial counting, modern poker games such as **Texas Hold'em**, **Omaha**, and **Seven-Card Stud** use slightly different dealing rules. The mathematics remains the same: we count combinations of cards that make a certain hand out of a larger set of cards available to each player.

Texas Hold'em

Texas Hold'em

Each player receives two private (*hole*) cards, and then five community cards are placed face-up on the table. A player's best five-card hand is chosen from those seven cards (2 personal + 5 community).

Total combinations. Each player can be dealt $\binom{52}{2} = 1{,}326$ possible hole-card combinations. If all five community cards are revealed, the total number of distinct seven-card sets is $\binom{52}{7} = 133{,}784{,}560$.

Example 1: Probability of being dealt a pair. Choose a rank for the pair (13 options) and two suits for that rank ($\binom{4}{2} = 6$); divide by all hole-card combinations:

$$P(\text{pair in starting hand}) = \frac{13\binom{4}{2}}{\binom{52}{2}} = \frac{78}{1{,}326} \approx 5.88\%.$$

Example 2: Probability of a flush from 7 cards. We call a hand a *flush* if at least five cards share a suit. Count first *including* straight flushes; then subtract those to get the usual "flush (non-straight)" category.

$$\text{Total 7-card hands} = \binom{52}{7} = 133{,}784{,}560.$$

Flushes including straight flushes. Choose a suit (4 ways) and take $k = 5, 6, 7$ cards from its 13 ranks; take the rest from the other 39 cards:

$$N_{\geq 5 \text{ same suit}} = 4\left[\binom{13}{5}\binom{39}{2} + \binom{13}{6}\binom{39}{1} + \binom{13}{7}\binom{39}{0}\right] = 4{,}089{,}228.$$

Straight flushes (including royals). The exact number of 7-card hands that contain at least one 5-card straight flush is much harder to calculate — we will omit the reasoning here since it is above the level of this class.

$$N_{\text{SF}} = 41{,}584 \quad \text{(this includes } 4{,}324 \text{ royals and } 37{,}260 \text{ other straight flushes).}$$

Flushes (non-straight).

$$N_{\text{flush, non-SF}} = N_{\geq 5 \text{ same suit}} - N_{\text{SF}} = 4{,}089{,}228 - 41{,}584 = 4{,}047{,}644.$$

Probabilities. Therefore

$$P(\text{flush including SF}) = \frac{4{,}089{,}228}{133{,}784{,}560} \approx 0.0305658 \;=\; 3.0566\%,$$

$$P(\text{flush (non-SF)}) = \frac{4{,}047{,}644}{133{,}784{,}560} \approx 0.0302549 \;=\; 3.0255\%.$$

For comparison, in five-card draw:

$$P(\text{flush including SF}) = \frac{4\binom{13}{5}}{\binom{52}{5}} = \frac{5{,}148}{2{,}598{,}960} \approx 0.198\%.$$

$$P(\text{flush (non-SF)}) = \frac{4\binom{13}{5} - 40}{\binom{52}{5}} = \frac{5{,}108}{2{,}598{,}960} \approx 0.1965\%.$$

Hand (best of 7)	**Count**	**Probability**
Royal flush	4,324	0.0000323%
Straight flush (non-royal)	37,260	0.0028%
Four of a kind	224,848	0.168%
Full house	3,473,184	2.60%
Flush	4,047,644	3.03%
Straight	6,180,020	4.62%
Three of a kind	6,461,620	4.83%
Two pairs	31,433,400	23.5%
One pair	58,627,800	43.8%
Nothing (high card)	23,294,460	17.4%

Seven-Card Stud

Seven-Card Stud

Each player receives 7 cards, but unlike Hold'em there are no community cards. Each player's best 5-card hand is chosen from their own 7 cards.

Total hands. There are again $\binom{52}{7} = 133{,}784{,}560$ possible 7-card combinations, and the probabilities of hand types are the same as those in Hold'em's "best-of-7" table above.

Example (exact): Probability of a full house in 7 cards. A 7-card hand is scored by its *best 5-card hand*. We will count how many 7-card sets have *best hand* = *full house*, then divide by $\binom{52}{7}$.

Rank–multiplicity patterns that allow a full house. Write a 7-card multiset of ranks by sorted multiplicities (e.g., 3–2–1–1). A full house (three of one rank and two of another) can occur within the following patterns:

$$\boxed{3-2-1-1, \quad 3-2-2, \quad 3-3-1, \quad 4-2-1, \quad 4-3}$$

However, any hand containing a four of a kind (patterns $4-2-1$ and $4-3$) has *best hand* = *four of a kind*, which outranks a full house. So for "best-hand full house," we must *exclude* those two patterns.

Counts by pattern (all suit choices included):

$$(A)\ 3\!-\!2\!-\!1\!-\!1: \quad \underbrace{13}_{\text{triple rank}}\binom{4}{3}\cdot\underbrace{12}_{\text{pair rank}}\binom{4}{2}\cdot\underbrace{\binom{11}{2}}_{\text{two singleton ranks}}\cdot 4^2 = 3{,}294{,}720.$$

$$(B)\ 3\!-\!2\!-\!2: \quad \underbrace{13}_{\text{triple rank}}\binom{4}{3}\cdot\underbrace{\binom{12}{2}}_{\text{two pair ranks}}\cdot\left(\binom{4}{2}\right)^2 = 123{,}552.$$

$$(C)\ 3\!-\!3\!-\!1: \quad \underbrace{\binom{13}{2}}_{\text{two triple ranks}}\cdot\left(\binom{4}{3}\right)^2\cdot\underbrace{11}_{\text{singleton rank}}\cdot\underbrace{4}_{\text{singleton suit}} = 54{,}912.$$

$$(D)\ 4\!-\!2\!-\!1: \quad \underbrace{13}_{\text{quad rank}}\cdot\underbrace{12}_{\text{pair rank}}\binom{4}{2}\cdot\underbrace{11}_{\text{singleton rank}}\cdot\underbrace{4}_{\text{singleton suit}} = 41{,}184.$$

$$(E)\ 4\!-\!3: \quad \underbrace{13}_{\text{quad rank}}\cdot\underbrace{12}_{\text{triple rank}}\cdot\binom{4}{3} = 624.$$

Hands whose *best* 5-card hand is a full house. Exclude patterns with a four of a kind:

$$N_{\text{best=FH}} = (A) + (B) + (C) = 3{,}294{,}720 + 123{,}552 + 54{,}912 = \boxed{3{,}473{,}184}.$$

Probability. With $\binom{52}{7} = 133{,}784{,}560$ total 7-card hands,

$$P(\text{full house in 7, as best hand}) = \frac{3{,}473{,}184}{133{,}784{,}560} \approx \boxed{2.597\%}.$$

Remark (if you count "contains a full house" regardless of best hand). If you instead count all 7-card hands that *contain* some 5-card full house (even if a stronger hand is present), you would *include* (D) and (E):

$$(A) + (B) + (C) + (D) + (E) = 3{,}514{,}992.$$

But the standard 7-card hand-type table partitions by *best* hand, hence uses $3{,}473{,}184$ for "full house."

Omaha Hold'em

Omaha Hold'em

Each player gets four hole cards, but must use *exactly two* of them together with three of the five community cards.

This rule changes combinatorial counts dramatically and requires mixed binomial reasoning (choose which 2 from 4, which 3 from 5, etc.). For instance:

$$\text{Ways to choose final hand} = \binom{4}{2}\binom{5}{3} = 60.$$

The sample space of starting hands is $\binom{52}{4} = 270{,}725$.

Video Poker and Simplified Draw Games

In one-player video poker (5-card draw with automatic scoring), the same $\binom{52}{5}$ sample space is used. For example:

$$P(\text{royal flush}) = \frac{4}{2{,}598{,}960} = 0.000154\%, \quad P(\text{straight flush}) = \frac{36}{2{,}598{,}960} = 0.00139\%.$$

Casino pay tables are built from these exact probabilities.

Observation. Adding more available cards (as in Hold'em or Stud) increases the probability of getting strong hands, because players can select the best 5 out of 7 instead of being limited to 5 total.

Different games — same mathematics: combinatorics, symmetry, and probability!

Homework

1. **Suit composition.**
   (a) How many 5-card hands contain all cards of different suits?
   (b) How many 5-card hands contain exactly 3 cards of one suit and 2 of another?
2. **Red and black.** How many 5-card hands contain exactly 3 red cards and 2 black cards?
3. **Containing a specific card.**
   (a) How many 5-card hands contain the Ace of Spades?
   (b) What is the probability that a 5-card hand has exactly one Ace?
   (c) What is the probability that a random 5-card hand contains at least one Ace?
4. **At least one face card.** Compute the probability that a 5-card hand has at least one Jack, Queen, or King.
5. **All same color but not one suit.** How many 5-card hands are all red but not all hearts or all diamonds?
6. **Flush draw.** Given that your 5-card hand already contains 4 hearts, what is the probability the next card is also a heart?
7. **Pair in two cards.** If you are dealt two cards, what is the probability they are of the same rank?
8. **Conditional on one card.** Given that your hand contains $K\heartsuit$, what is the probability you have at least one more heart?
9. **One-card game.** Two players each draw one card. What is the probability Player 1's card outranks Player 2's?
10. **Suit patterns.** How many distinct suit patterns exist for a 5-card hand?
11. **Comparing pairs.** You and a friend each have exactly one pair. What is the probability your pair is higher?