MATH 7: HANDOUT 9

PROBABILITY REFRESHER. BINOMIAL PROBABILITIES

Probability Refresher

What Is Probability?

In mathematics, **probability** measures how likely an event is to happen. For any event A, its probability $P(A)$ satisfies $0 \le P(A) \le 1$.

- $P(A) = 1$ means A happens *with certainty in the model* (also called "almost surely" in advanced courses).
- $P(A) = 0$ means A happens *with probability zero in the model*.

In **finite, equally likely** settings (like a fair die or a deck of cards), $P(A) = 0$ indeed means A is impossible, and $P(A) = 1$ means A is certain.

In **continuous** settings, events with probability 0 can still occur. For example, if a real number is chosen uniformly from $[0, 1]$, the event "the number equals $1/2$" has $P = 0$ but is not impossible.

If all outcomes are equally likely (finite sample space), the **classical definition** applies:

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{number of all outcomes}}.$$

This rule is called the **classical definition of probability**.

Sample Space and Events

The set of all possible outcomes is called the **sample space** and is denoted by Ω. Each individual outcome is called a **sample point**.

An **event** is any subset of the sample space Ω.

When rolling a six-sided die:

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

The event "rolling an even number" is

$$A = \{2, 4, 6\}.$$

Thus, $P(A) = \frac{3}{6} = \frac{1}{2}$.

Rules of Probability

For any events A and B in the same sample space:

1. $P(\Omega) = 1$.
2. $P(\varnothing) = 0$.
3. $P(\text{not } A) = 1 - P(A)$.
4. $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

   Explanation: When we add $P(A)$ and $P(B)$, the outcomes that belong to both A and B are counted *twice* — once in $P(A)$ and once in $P(B)$. To fix that, we subtract the overlap $P(A \text{ and } B)$ once.

   Example: In a class of 30 students:

- A = event that a randomly selected student plays soccer,
- B = event that a randomly selected student plays basketball.

Suppose:

$$P(A) = \frac{12}{30}, \quad P(B) = \frac{10}{30}, \quad P(A \text{ and } B) = \frac{4}{30}.$$

Then

$$P(A \text{ or } B) = \frac{12}{30} + \frac{10}{30} - \frac{4}{30} = \frac{18}{30}.$$

This means that the probability that randomly selected plays either soccer, basketball, or both is $\frac{18}{30}$.

5. If A and B are **mutually exclusive** (they cannot happen at the same time), then:

$$P(A \text{ or } B) = P(A) + P(B).$$

Independent Events

Two events A and B are called **independent** if the occurrence of one does not affect the other. In that case,

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

You toss a fair coin and roll a die.

$$P(\text{Head and } 6) = P(\text{Head}) \times P(6) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}.$$

Experiments with Repeated Trials

When the same experiment is repeated several times, the results of the individual trials can be combined.

If the probability of getting a Head in one coin toss is $1/2$, then the probability of getting two Heads in a row is:

$$P(\text{HH}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

In the next section, we will learn how to calculate the probability of getting a specific number of successes (for example, exactly k Heads in n tosses). To do that, we will use **binomial coefficients**.

Probabilty Inttoduction

- **Probability** measures how likely an event is: $0 \leq P(A) \leq 1$.
- **Sample space** Ω: the set of all possible outcomes.
- **Event**: any subset of Ω;
  its probability is $P(A) = \dfrac{\text{favorable outcomes}}{\text{all outcomes}}$ (if all outcomes are equally likely).
- **Basic rules:**

$$P(\Omega) = 1, \quad P(\varnothing) = 0, \quad P(\text{not } A) = 1 - P(A),$$
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

- If A and B are **mutually exclusive**: $P(A \text{ or } B) = P(A) + P(B)$.
- If A and B are **independent**: $P(A \text{ and } B) = P(A) \cdot P(B)$.

Motivation: Counting Probabilities in Repeated Trials

Before we introduce the general binomial formula, let us build intuition by looking carefully at simple repeated experiments. We will start with coin tosses, then move on to rolling a die, and finally generalize.

Example 1: Tossing a Coin 5 Times

Suppose we toss a fair coin 5 times. Each toss has two equally likely outcomes:

$$p = P(\text{Head}) = \tfrac{1}{2}, \qquad q = P(\text{Tail}) = \tfrac{1}{2}.$$

We want to compute each of the probabilities:

$$P(0 \text{ Heads}),\ P(1 \text{ Head}),\ \ldots,\ P(5 \text{ Heads}).$$

0 Heads. To get 0 Heads, all five tosses must be Tails:

$$TTTTT.$$

There is only one such sequence, and its probability is

$$P(0H) = \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32}.$$

1 Head. We need exactly 1 Head. There are $\binom{5}{1} = 5$ possible sequences (choose the position of the Head). Each sequence has probability $(\frac{1}{2})^5$. Therefore,

$$P(1H) = 5 \cdot \left(\tfrac{1}{2}\right)^5 = \tfrac{5}{32}.$$

2 Heads. We choose 2 positions out of 5:

$$\binom{5}{2} = 10.$$

Each sequence again has probability $(\frac{1}{2})^5$, so

$$P(2H) = 10 \cdot \tfrac{1}{32} = \tfrac{10}{32}.$$

In general for k Heads.

$$P(k \text{ Heads in } 5 \text{ tosses}) = \binom{5}{k} \left(\tfrac{1}{2}\right)^5.$$

Example 2: Rolling a Die 4 Times (Success = 6)

Now consider a different experiment. We roll a die 4 times. Let

$$p = P(\text{rolling a 6}) = \tfrac{1}{6}, \qquad q = P(\text{not a 6}) = \tfrac{5}{6}.$$

We compute:

$$P(0 \text{ sixes}),\ P(1 \text{ six}),\ P(2 \text{ sixes}),\ P(3 \text{ sixes}),\ P(4 \text{ sixes}).$$

0 sixes. All four rolls must be failures:

$$P(0 \text{ sixes}) = \left(\tfrac{5}{6}\right)^4.$$

1 six. There are $\binom{4}{1} = 4$ ways to choose which roll is a 6. Each sequence has probability $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3$, so

$$P(1 \text{ six}) = 4 \left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)^3.$$

2 sixes. Choose 2 rolls out of 4:

$$\binom{4}{2} = 6.$$

So

$$P(2 \text{ sixes}) = 6\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)^2.$$

And similarly for 3 or 4 sixes. Each case follows the same pattern: choose the positions of the successes, and multiply the probabilities.

Observing the Pattern

Both examples follow the same structure:

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{\,n-k}.$$

This is exactly the **binomial probability formula**. We now present it in full generality.

Binomial Probabilities

In many real-life experiments, we repeat the same random trial several times under identical conditions. Each trial can result in either a **success** or a **failure**.

For example:

- tossing a coin (success = Head),
- rolling a die (success = getting a 6),
- shooting a basketball (success = making the shot),
- guessing a multiple-choice question (success = correct answer).

When each trial is independent (that is, one outcome does not affect the others), and the probability of success in each trial is the same, such an experiment is called a **binomial experiment**.

Terminology

Trial One instance of an experiment (for example, one coin toss).

n Number of trials.

Success The desired outcome in a trial (e.g., getting a Head, hitting a target).

k Number of successes we want to get.

p Probability of success in a single trial.

Failure Any trial that does not result in success.

q Probability of failure, $q = 1 - p$.

The Binomial Probability Formula

If we want to find the probability of getting exactly k successes in n independent trials, where the probability of success in each trial is p, then

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k},$$

where:

- $\binom{n}{k}$ = number of ways to choose which k trials are successful,
- p^k = probability that those k trials succeed,
- q^{n-k} = probability that the remaining $n-k$ trials fail.

Explanation: We multiply $p^k q^{n-k}$ because the outcomes of individual trials are independent, and we multiply by $\binom{n}{k}$ because there are $\binom{n}{k}$ possible arrangements of which trials succeed.

Examples

Coin Tosses

You toss a fair coin 5 times. What is the probability of getting exactly 2 Heads?

Here $n = 5$, $k = 2$, $p = \frac{1}{2}$, $q = \frac{1}{2}$.

$$P = \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = 10 \times \frac{1}{32} = \frac{10}{32} = \tfrac{5}{16}.$$

So the probability of getting exactly two Heads is $\frac{5}{16}$.

Rolling a Die

You roll a six-sided die 6 times. What is the probability of getting a 6 exactly once?

Here $n = 6$, $k = 1$, $p = \frac{1}{6}$, $q = \frac{5}{6}$.

$$P = \binom{6}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = 6 \cdot \frac{5^5}{6^6} = \frac{5^5}{6^5} \cdot \frac{1}{6}.$$

So there is about a 40% chance of rolling a 6 at least once in 6 tries, and about 33% chance of rolling it *exactly once*.

Basketball Free Throws

A basketball player makes each free throw with probability $p = 0.8$. She takes $n = 5$ free throws. What is the probability that she makes exactly 4 of them?

We have $q = 1 - p = 0.2$ and $k = 4$.

$$P = \binom{5}{4} (0.8)^4 (0.2)^1 = 5 \times 0.4096 \times 0.2 = 0.4096.$$

So there is about a 41% chance that she will make exactly four out of five shots.

At Least One Success

Sometimes we want the probability of getting *at least one* success, not an exact number.
If the probability of success in one trial is p, then

$$P(\text{at least one success}) = 1 - P(\text{no successes}) = 1 - q^n.$$

Example: You roll a die 6 times. What is the probability of getting at least one six?

$$P = 1 - \left(\frac{5}{6}\right)^6 \approx 1 - 0.3349 = 0.6651.$$

So there is about a 66.5% chance that you will roll at least one six in six rolls.

Interpreting Binomial Probabilities

- The probabilities for $k = 0, 1, 2, \ldots, n$ always add up to 1.
- The most likely number of successes (the *mode*) is usually near $n \cdot p$.
- For large n, the distribution of possible k values forms a bell-like shape centered at np.

The binomial model helps describe many real-world processes: games of chance, genetics, manufacturing defects, surveys, and much more.

Key Facts about Binomial Probability

- A **binomial experiment** consists of n independent trials, each with two outcomes: *success* (probability p) and *failure* (probability $q = 1 - p$).
- The probability of getting exactly k successes in n trials is

$$P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

- $\binom{n}{k}$ counts the number of ways to choose which k trials succeed.
- The probabilities for all $k = 0, 1, \ldots, n$ add up to 1.
- Probability of **at least one success:**

$$P(X \geq 1) = 1 - q^n.$$

- The most likely number of successes is usually near $n \cdot p$.

Discussion Problem: Fair Games

In real life, probability is not just about rolling dice or flipping coins. It shows up whenever someone designs a game, a lottery, or even an insurance plan. If you go to a fair and pay to throw balls at a target, the organizers have to decide:

- How likely is it that players will win?
- How much money will they pay out in prizes?
- How much should they charge so they do not lose money in the long run?

To answer these questions, we need two ideas:

- **Probabilities of different outcomes** (for example, getting 0, 1, 2, or 3 hits).
- **Expected value**, which tells us what happens *on average* if we repeat the game many times.

In this problem we will analyze a simple game and see how probability and expected value help us decide whether the game is a good deal for the player and for the organizers.

Fair Game Problem

Assume that each ball hits the target with probability p (the same for every ball), and misses with probability $q = 1 - p$. The throws are independent. You win a prize if you get at least one hit.

(a) If you are given 3 balls, find the probabilities that *all* 3 *are hits*, *all* 3 *are misses*, there is *exactly* 1 *hit*, there are *exactly* 2 *hits*, and there is *at least one hit*.

(b) If the prize costs the organizers about \$$W$, how much should they charge for 3 balls to *break even*? (Hint: the game should bring in the same money it gives away on average.)

(c) If you could play hundreds of times, how much money would you "expect to" win or lose per game? What happens if the price is higher or lower than your answer in part (b)?

(d) Discuss: why might the organizers choose to charge a bit more than the break-even price? Is this fair? Why or why not?

Solution and Discussion

(a) Probabilities for 3 balls.

Each ball is a "success" (hit, H) with probability p and a "failure" (miss, M) with probability $q = 1 - p$.

- **All 3 are hits.** For all 3 balls to hit, we need

$$(\text{hit}) \cdot (\text{hit}) \cdot (\text{hit}),$$

so the probability is

$$P(3 \text{ hits}) = p \cdot p \cdot p = p^3.$$

- **All 3 are misses.** For all 3 balls to miss, we need

$$(\text{miss}) \cdot (\text{miss}) \cdot (\text{miss}),$$

so the probability is

$$P(3 \text{ misses}) = q \cdot q \cdot q = q^3 = (1 - p)^3.$$

- **Exactly 1 hit.** There are 3 different ways to choose which throw is the only hit:

$$(\text{HMM}),\ (\text{MHM}),\ (\text{MMH}).$$

Each has probability pq^2, so altogether:

$$P(\text{exactly 1 hit}) = 3pq^2 = 3p(1 - p)^2.$$

- **Exactly 2 hits.** There are two good ways to think about this:

  Approach 1: Choose which two throws are the *hits*. We must pick which 2 out of the 3 throws hit the target:

  $$\binom{3}{2} = 3 \text{ ways.}$$

  The patterns are

  $$(\text{HHM}),\ (\text{HMH}),\ (\text{MHH}),$$

  and each one has probability $p^2 q$. So

  $$P(\text{exactly 2 hits}) = \binom{3}{2} p^2 q = 3p^2 q = 3p^2(1-p).$$

  Approach 2: Choose which throw is the *miss*. Instead of choosing the 2 hits, we can choose the 1 miss. There are again

  $$\binom{3}{1} = 3$$

  choices for where the miss goes, and every pattern still has probability $p^2 q$. So we get the same result:

  $$P(\text{exactly 2 hits}) = 3p^2 q.$$

- **At least one hit.**

  There are two common ways to think about this:

  Approach 1: Direct (but messy) counting. At least one hit means:

  $$1 \text{ hit} \;\text{ or }\; 2 \text{ hits} \;\text{ or }\; 3 \text{ hits}.$$

  We could compute

  $$P(\text{at least one hit}) = P(1 \text{ hit}) + P(2 \text{ hits}) + P(3 \text{ hits}),$$

  where

  $$P(1 \text{ hit}) = \binom{3}{1} p^1 q^2, \quad P(2 \text{ hits}) = \binom{3}{2} p^2 q^1, \quad P(3 \text{ hits}) = \binom{3}{3} p^3 q^0,$$

  so

  $$P(\text{at least one hit}) = 3pq^2 + 3p^2 q + p^3.$$

  This works, but it is a bit long, and if you have more than 3 trials, it will take even longer.

  Approach 2: Using the complement (much easier). The only way to *not* get at least one hit is to get *zero* hits (all misses). So:

  $$P(\text{at least one hit}) = 1 - P(\text{no hits}) = 1 - P(3 \text{ misses}) = 1 - q^3 = 1 - (1-p)^3.$$

  So the answers are:

  $$P(3 \text{ hits}) = p^3, \qquad P(3 \text{ misses}) = (1-p)^3,$$
  $$P(1 \text{ hit}) = 3p(1-p)^2, \qquad P(2 \text{ hits}) = 3p^2(1-p),$$
  $$P(\text{at least one hit}) = 1 - (1-p)^3.$$

  Intuitively: if p is small, then $(1-p)^3$ is close to 1, so $1-(1-p)^3$ is small and you rarely win. If p is large, then $(1-p)^3$ is small, and $1-(1-p)^3$ is close to 1, so you almost always win.

Numerical Example (using $p = 0.25$)

If each ball hits with probability 0.25, then

$$P(3\text{ hits}) = 0.25^3 = 0.015625 \approx 1.56\%.$$

Hitting all 3 is very rare.

$$P(3\text{ misses}) = 0.75^3 = 0.421875 \approx 42.19\%.$$

For exactly one hit, we use $3pq^2$:

$$P(\text{exactly 1 hit}) = 3(0.25)(0.75^2) = 3(0.25)(0.5625) = 0.421875.$$

For exactly two hits, we use $3p^2q$:

$$P(\text{exactly 2 hits}) = 3(0.25^2)(0.75) = 3(0.0625)(0.75) = 0.140625.$$

Finally, the probability of getting at least one hit is

$$P(\text{at least one hit}) = 1 - 0.75^3 = 0.578125.$$

So with $p = 0.25$, the player wins a prize about **57.8%** of the time.

(b) Break-even price for the organizers.

Assume the organizers give out a prize *only* when the player gets at least one hit. Each prize costs the organizers about $\$W$.

Let x be the price (in dollars) they charge for 3 balls (one game).

- In every game, they *collect* exactly $\$x$ from the player.
- They *pay out* $\$W$ only in games where the player wins (gets at least one hit).

From part a), the probability that a player wins is

$$P(\text{win}) = P(\text{at least one hit}) = 1 - (1-p)^3.$$

If many, many people play, about this fraction of games will produce a win. Among, say, N games, you would expect roughly

$$\text{number of wins} \approx N \cdot \big(1 - (1-p)^3\big).$$

So the total *payout* for N games is approximately

$$W \times N \cdot \big(1 - (1-p)^3\big).$$

The total *income* for N games is

$$N \cdot x.$$

Break even means:

$$\text{total income} = \text{total payout}.$$

So we set

$$Nx = W \cdot N \cdot \big(1 - (1-p)^3\big).$$
$$x = W \cdot \big(1 - (1-p)^3\big).$$

Break-even price:

$$\boxed{x = W\big(1 - (1-p)^3\big).}$$

This is the price at which, in the long run, the organizers neither gain nor lose money overall.

Numerical Example (using $p = 0.25$ and $W = \$5$)

We now assume $p = 0.25$: each ball hits the target with probability 0.25 and misses with probability 0.75, and the prize costs \$5.
From part a), the probability of getting *at least one hit* with 3 balls is

$$P(\text{win}) = P(\text{at least one hit}) = 1 - P(\text{3 misses}) = 1 - 0.75^3 = 1 - 0.421875 = 0.578125.$$

This means that if you play this game a very large number of times, you will win in about 57.8125% of the games and lose in about 42.1875% of the games.

Now let us think like the organizers.

- They give a prize worth \$5 whenever the player wins (gets at least one hit).
- They give nothing when the player loses.
- They charge $\$x$ for each game (3 balls).

Imagine that the game is played N times (where N is very large, like $N = 1000$ or $N = 10{,}000$). Then approximately:

$$\text{number of wins} \approx N \cdot 0.578125, \quad \text{number of losses} \approx N \cdot 0.421875.$$

Total money collected from players is:

$$\text{Income} \approx N \cdot x.$$

Total money paid out in prizes is:

$$\text{Payout} \approx (\text{number of wins}) \cdot 5 \approx N \cdot 0.578125 \cdot 5 = N \cdot 2.890625.$$

To *break even*, the organizers want

$$\text{Income} = \text{Payout}.$$

So we set

$$N \cdot x = N \cdot 2.890625.$$

Now divide both sides by N:

$$x = 2.890625.$$

So the break-even price is

$$\boxed{x \approx \$2.89.}$$

In words: on average, each game costs the organizers about \$2.89 in prize money (because there is only a 57.8% chance that they actually pay the \$5). If they charge exactly \$2.89, then over many games they neither make nor lose money overall.
If they charge:

- less than \$2.89, they lose money in the long run,
- more than \$2.89, they make money in the long run.

(c) Long-run gain or loss per game.

Now imagine you personally play this game hundreds or thousands of times. Of course, in any one particular game, you might win or lose money. But over many games, there is a *typical* average gain (or loss) per game.

Let:

$$P(\text{win}) = 1 - (1-p)^3, \quad P(\text{lose}) = (1-p)^3.$$

Suppose the price per game is x dollars. From the *player's* point of view:

- If you **win**, you get a prize worth $\$W$, but you paid \$x to play. Your net change in money for that game is

$$+W - x.$$

- If you **lose**, you get no prize, but still paid \$x. Your net change is

$$-x.$$

Now imagine playing, say, N games (with N very large).

- About $N \cdot P(\text{win})$ of those games will be wins.
- About $N \cdot P(\text{lose})$ of those games will be losses.

Total net gain after N games is approximately

$$(\text{number of wins}) \cdot (W - x) + (\text{number of losses}) \cdot (-x).$$

So

$$\text{Total net gain} \approx N \cdot P(\text{win}) \cdot (W - x) + N \cdot P(\text{lose}) \cdot (-x).$$

Divide by N to get the **average gain per game**:

$$\text{Average gain per game} \approx P(\text{win})\,(W - x) + P(\text{lose})\,(-x).$$

Now substitute $P(\text{lose}) = 1 - P(\text{win})$:

$$\begin{aligned} \text{Average gain per game} &= P(\text{win})\,(W - x) + (1 - P(\text{win}))\,(-x) \\ &= W \cdot P(\text{win}) - x\,P(\text{win}) - x + x\,P(\text{win}) \\ &= W \cdot P(\text{win}) - x. \end{aligned}$$

So the player's long-run average gain per game is

$$\boxed{\text{Average gain per game} = W\big(1 - (1-p)^3\big) - x.}$$

- If $x = W\big(1 - (1-p)^3\big)$ (the break-even price from part b)), then the average gain per game is 0. In the long run, you neither gain nor lose money.
- If x is **higher** than this amount, the average gain per game is **negative**: over many games, you lose money on average.
- If x is **lower** than this amount, the average gain per game is **positive**: over many games, you come out ahead on average.

So this average gain per game is the number you "should expect" as your typical profit (or loss) for one game when you play a very large number of times.

Numerical Example (using $p = 0.25$ and $W = \$5$)

Now we look at the game from the *player's* point of view. We already know:

$$P(\text{win}) = 0.578125, \quad P(\text{lose}) = 0.421875.$$

Let the price per game be $\$x$. In a single game:

- If the player **wins**, they receive a prize worth \$5, but they have paid $\$x$ to play. So their net change is

$$\text{gain in a win} = 5 - x.$$

- If the player **loses**, they get no prize and still pay $\$x$. So their net change is

$$\text{gain in a loss} = -x.$$

Now imagine the player plays this game N times (with N very large). We can approximate:

$$\text{number of wins} \approx N \cdot 0.578125, \quad \text{number of losses} \approx N \cdot 0.421875.$$

The **total net gain** after N games is approximately:

$$\text{Total gain} \approx (\text{wins}) \cdot (5 - x) + (\text{losses}) \cdot (-x)$$

$$\approx N \cdot 0.578125 \cdot (5 - x) + N \cdot 0.421875 \cdot (-x).$$

Factor out N:

$$\text{Total gain} \approx N \left[0.578125(5 - x) + 0.421875(-x)\right].$$

To find the *average gain per game*, divide by N:

$$\text{Average gain per game} \approx 0.578125(5 - x) + 0.421875(-x).$$

Now simplify this expression step by step:

$$\begin{aligned}\text{Average gain per game} &= 0.578125 \cdot 5 \; - \; 0.578125 \cdot x \; - \; 0.421875 \cdot x \\ &= 2.890625 \; - \; (0.578125 + 0.421875)x \\ &= 2.890625 - x.\end{aligned}$$

So the player's long-run average gain per game is

$$\boxed{\text{Average gain per game} = 2.890625 - x.}$$

This single number tells you what typically happens if you play many, many times:

- If $x = 2.89$ (the break-even price from part (b)), then

  $$\text{Average gain per game} \approx 2.890625 - 2.890625 = 0.$$

  Over many games, you neither gain nor lose money overall.

- If $x > 2.89$, then $2.890625 - x$ is negative. For example, if $x = 3$:

  $$\text{Average gain per game} = 2.890625 - 3 = -0.109375.$$

  So you lose about 11 cents per game on average.

- If $x < 2.89$, then $2.890625 - x$ is positive. For example, if $x = 2.50$:

  $$\text{Average gain per game} = 2.890625 - 2.50 = 0.390625.$$

  So you win about 39 cents per game on average.

In a single play, anything can happen: you might win or lose a few dollars. But over many, many plays, your average gain per game will settle close to $2.890625 - x$. This "typical" gain or loss per game is the key quantity we are trying to understand.

(d) Why might the organizers charge more than the break-even price?

Some reasons organizers might charge a bit more:

- They need to pay for costs other than the prize (salaries, rent, equipment, decorations, etc.).
- They want to make a profit to keep the event running or to raise money (for a fair, school, charity, etc.).
- There is randomness: even at the theoretical break-even price, in real life they might have a streak where many players win at once. Charging a bit more gives them a safety margin.

Is this fair?

- **One point of view:** It is fair if players understand the rules and the price, and are willing to pay for the fun of playing, knowing that on average they will lose a little.
- **Another point of view:** If the price is much larger than the break-even value, some may feel the game is "rigged" in favor of the organizers.

This is exactly how many real-world games work: *lotteries, casinos, carnival games, and even some insurance contracts* are designed so that, on average, one side comes out ahead over many repetitions, even though any single outcome may look lucky or unlucky.

Numerical Insight (using $p = 0.25$)

When $p = 0.25$, winning is not easy: even with 3 balls, the chance of winning is only about 57.8%. If the organizers charge the break-even price of $2.89, their long-term profit is zero.
But real-life organizers face variability: in a short fair, a streak of lucky players could wipe out their profit. Charging slightly more, say $3.00, gives them a built-in safety margin:

$$\text{Average gain for the player} = 2.890625 - 3.00 = -0.109375.$$

So the player loses about 11 cents per game on average — still small enough that the game feels fun and fair.

Practice Problems

1. A fair die is rolled twice. Find:
   (a) $P(\text{sum of } 7)$,
   (b) $P(\text{sum greater than } 9)$.

2. A bag contains 3 red, 2 green, and 5 blue marbles.
   (a) What is the probability of picking a green marble?
   (b) What is the probability of picking a marble that is not blue?

3. Two fair dice are rolled.
   (a) Write the sample space Ω.
   (b) Let $A =$ "the sum is even," and $B =$ "the first die shows an even number." List the outcomes in each event.
   (c) Find $P(A)$, $P(B)$, and $P(A \text{ and } B)$. Are A and B independent? Explain your reasoning.

4. Two independent spinners each have equal sectors labeled $1, 2, 3, 4$.
   (a) Write the sample space for all possible pairs of outcomes (A, B).
   (b) Find the probability that both spinners show an even number.
   (c) Find the probability that the two numbers add up to 6.
   (d) Are the events "both even" and "sum = 6" independent? Compute and check $P(A \text{ and } B) = P(A)P(B)$.

5. A fair coin is tossed 6 times. Find the probability of getting:
   (a) exactly 3 Heads,
   (b) at least 4 Heads.

6. A student guesses on a 5-question true/false quiz. What is the probability that the student answers exactly 3 questions correctly?

7. You roll a fair six-sided die. You pay $1 to play no matter what. If you roll a 6, you win $5; otherwise you win nothing.
   (a) What is the probability of winning (rolling a 6)?
   (b) If you play once, what are the two possible *net* outcomes (profit/loss)?
   (c) How much should the entry fee be so that the game is *fair* (average net gain = 0)?

8. A student practices throwing a bean bag at a target. Each throw hits with probability 0.3. Every hit earns $2; every miss loses $1 (for paying to play).
   (a) What is the chance of a hit or miss?
   (b) If you play once, what could happen?
   (c) If you play many times, do you expect to come out ahead or behind?
   (d) Try to find the average gain or loss per throw. (Hint: multiply each possible gain by its probability.)

Practice Problems: Solutions

1. **Die rolled twice.**
   (a) $P(\text{sum } 7) = \dfrac{6}{36} = \dfrac{1}{6}$.
   (b) Sums > 9 are $10, 11, 12$ with counts $3, 2, 1$ (total 6 outcomes), so $P(\text{sum} > 9) = \dfrac{6}{36} = \dfrac{1}{6}$.
2. **Bag:** $3R, 2G, 5B$ **(total** 10**).**
   (a) $P(G) = \dfrac{2}{10} = \dfrac{1}{5}$.
   (b) $P(\text{not blue}) = \dfrac{3+2}{10} = \dfrac{5}{10} = \dfrac{1}{2}$.
3. **Two fair dice.**
   (a) Ω has 36 ordered outcomes (i, j), $i, j \in \{1, \ldots, 6\}$.
   (b) $A = \{\text{sum even}\}$ (both even or both odd); $B = \{\text{first die even}\}$.
   (c) $P(A) = \dfrac{18}{36} = \dfrac{1}{2}$, $\quad P(B) = \dfrac{18}{36} = \dfrac{1}{2}$, $\quad P(A \cap B) = \dfrac{9}{36} = \dfrac{1}{4}$ (first even and second even). Since $P(A \cap B) = P(A)P(B)$, events A and B are **independent**.
4. **Two independent spinners (1–4).**
   (a) $|\Omega| = 16$ ordered pairs.
   (b) $P(\text{both even}) = \left(\dfrac{2}{4}\right)\left(\dfrac{2}{4}\right) = \dfrac{1}{4}$.
   (c) Sum $= 6$ outcomes: $(2, 4), (3, 3), (4, 2) \Rightarrow P = \dfrac{3}{16}$.
   (d) $P(\text{both even} \cap \text{sum } 6) = \dfrac{2}{16} = \dfrac{1}{8}$ (only $(2, 4), (4, 2)$). But $P(\text{both even})P(\text{sum } 6) = \dfrac{1}{4} \cdot \dfrac{3}{16} = \dfrac{3}{64}$. Since $\dfrac{1}{8} \neq \dfrac{3}{64}$, the events are **not independent**.
5. **Coin tossed 6 times.**
   (a) $P(\text{exactly 3 H}) = \dbinom{6}{3}\left(\dfrac{1}{2}\right)^6 = \dfrac{20}{64} = \dfrac{5}{16}$.
   (b) $P(\text{at least 4 H}) = \dfrac{\binom{6}{4} + \binom{6}{5} + \binom{6}{6}}{2^6} = \dfrac{15 + 6 + 1}{64} = \dfrac{22}{64} = \dfrac{11}{32}$.
6. **True/False quiz, 5 questions (guessing).**
$$P(\text{exactly 3 correct}) = \binom{5}{3}\left(\frac{1}{2}\right)^5 = \frac{10}{32} = \frac{5}{16}.$$
7. **Die game: win \$5 on a 6; otherwise pay \$1.**
   (a) The die has 6 equally likely faces, so
$$P(\text{win}) = \frac{1}{6}.$$
   (b) If you roll a 6, you win \$5 but have already paid \$1, so your net gain is $+4$. If you do not roll a 6, you win nothing and lose your entry fee, so your net outcome is -1.

(c) Let X be the net gain per play. Then on average per game you will be at

$$\frac{1}{6}(+4) + \frac{5}{6}(-1) = \frac{4}{6} - \frac{5}{6} = -\frac{1}{6}.$$

On average, a player loses about \$0.17 per game. To make the game **fair**, the expected gain must be zero.

Let the fair entry fee be c. Then

$$E(X) = \frac{1}{6}(5-c) + \frac{5}{6}(-c) = 0.$$

Simplifying:

$$\frac{5}{6} - c = 0 \quad \Rightarrow \quad c = \frac{5}{6} \approx \$0.83.$$

So if the entry fee were about \$0.83, the game would be fair.

8. **Bean bag: hit prob 0.3, $+\$2$ for hit, $-\$1$ for miss.**

(a) $P(\text{hit}) = 0.3,\ P(\text{miss}) = 0.7.$

(b) Outcomes: $+\$2$ (hit) or $-\$1$ (miss).

(c) Over many plays you expect a small loss (misses are more likely and cost money).

(d) Average gain per throw: $0.3 \cdot 2 + 0.7 \cdot (-1) = 0.6 - 0.7 = -\$0.10$ (lose 10 cents per throw on average).

Homework

1. A fair die is rolled 3 times. Find the probability that the sum of the numbers is at least 15.

2. In a multiple-choice test, each question has 4 options, only one correct. If a student guesses all answers, what is the probability of getting:

   (a) exactly 2 correct out of 5,

   (b) at least one correct out of 5?

3. A six-sided die is rolled once.

   (a) Write the sample space Ω.

   (b) Let $A =$ "rolling an even number," $B =$ "rolling a number greater than 4." List the outcomes in each event.

   (c) Find $P(A)$, $P(B)$, and $P(A \text{ or } B)$.

   (d) Are A and B independent? Explain your reasoning.

4. Two fair dice are rolled.

   (a) Let $A =$ "the sum is divisible by 3," and $B =$ "the first die shows a 2." Compute $P(A)$, $P(B)$, and $P(A \text{ and } B)$.

   (b) Are these events independent?

5. Two cards are drawn *without replacement* from a standard deck.

   (a) Find the probability that both are Aces.

   (b) Find the probability that both are Hearts.

   (c) Are the two draws independent now? Explain what changes.

6. In each of 4 races, the Democrats have a 60% chance of winning. Assuming the races are independent, find:

   (a) the probabilities of winning $0, 1, 2, 3,$ or all 4 races,

   (b) the probability of winning at least one race,

   (c) the probability of winning a majority of the races.

7. A (blindfolded) marksman hits the target 4 times out of 5 on average. If he fires 4 shots, find the probability of:

   (a) more than 2 hits,

   (b) at least 3 misses.

8. A fair game at a carnival involves tossing small balls into bottles. Each ball has a 20% chance of landing inside a bottle (that is, probability 0.2 of success). If at least one ball lands inside, you win a stuffed toy worth about $5.

   (a) If you are given 5 balls, find the probabilities that all 5 are hits, all 5 are misses, and at least one is a hit.

   (b) If the prize costs the organizers about $5, how much should they charge for 5 balls to *break even*?

   (c) If you could play hundreds of times, how much money would you *expect to win or lose per game*? What happens if the price is higher or lower than your answer in part (b)?

9. A box contains 3 red and 2 blue marbles. Two marbles are drawn *without replacement*. Find the probability that:

(a) both are red,
(b) at least one is blue.