

MATH 7: HANDOUT 8

INTRODUCTION TO COMBINATORICS

Introduction

Combinatorics is the branch of mathematics concerned with **counting, arranging, and selecting objects** according to specific rules. It deals with questions like: How many different ways can students be seated in a classroom? How many possible poker hands are there? How many routes can a traveler take across a grid of streets?

The origins of combinatorics go back thousands of years. Ancient Indian and Chinese mathematicians studied arrangements of numbers and basic counting principles. In the 9th century, the Persian mathematician al-Karaji wrote about binomial coefficients, which later appeared prominently in **Pascal's Triangle** in the 17th century. During the Renaissance, mathematicians such as Cardano applied combinatorial reasoning to problems in gambling and probability. Today, combinatorics is a central part of modern mathematics, with applications ranging from computer science and cryptography to biology and physics.

Counting

Counting is one of the central problems in combinatorics: given a set of objects, in how many different ways can we perform certain selections, arrangements, or orderings?

The Fundamental Principle of Counting (Multiplication Rule)

If the first task can be performed in m ways, and for each of these choices a second task can be performed in n ways, and for each such combination a third task can be performed in k ways, then the total number of ways to perform the sequence of tasks is:

$$m \cdot n \cdot k \cdot \dots$$

Example: Suppose you are ordering lunch. You can choose from 3 different sandwiches, 2 different side dishes, and 4 drinks. Then the number of possible meal combinations is:

$$3 \cdot 2 \cdot 4 = 24.$$

This principle underlies all the more advanced counting rules we will study.

Permutations (Order Matters)

A **permutation** is an arrangement of objects where order matters, and no repetitions are allowed.

1. Choosing first, second, and third place winners from a group of n participants can be done in

$$n \cdot (n - 1) \cdot (n - 2)$$

ways.

2. Arranging all n students in a line for a class photo can be done in

$$n!$$

ways, where *factorial* $n! = n \cdot (n - 1) \cdot \dots \cdot 2 \cdot 1$.

General Rule. If we choose and arrange k objects out of n (so that order matters and no repetition is allowed), the number of permutations is

$$P(n, k) = {}_n P_k = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

This formula reduces to $n!$ when $k = n$, since we are arranging all objects.

More examples.

- Listing n desserts in order of preference can be done in $n!$ ways.
- Assigning k different seats in a row to k out of n students can be done in

$$P(n, k) = {}_n P_k = \frac{n!}{(n-k)!}.$$

A Note on Factorials

The expression $n!$ (read “ n factorial”) is defined as

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

for any positive integer n . Factorials grow very quickly:

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120.$$

Why is $0! = 1$? At first it may seem strange, but there are several consistent reasons:

- **Combinatorial reason:** $n!$ counts the number of ways to arrange n distinct objects. If $n = 0$, there is exactly one way to “arrange” nothing at all — by doing nothing! So $0!$ must equal 1.
- **Permutation reason:** The number of ways to arrange k objects chosen from n is

$$P(n, k) = \frac{n!}{(n-k)!}.$$

If we take $k = n$, this becomes $P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!}$. But we already know that $P(n, n)$ should equal $n!$, since arranging all n objects simply means permuting them in every possible order. Therefore,

$$n! = \frac{n!}{0!} \implies 0! = 1.$$

- **Recursive reason:** Factorials satisfy the rule

$$n! = n \cdot (n-1)!.$$

For $n = 1$, this gives $1! = 1 \cdot 0!$. Since $1! = 1$, it follows that $0! = 1$.

Thus, defining $0! = 1$ makes sense from every perspective. It ensures that our counting rules and formulas remain consistent.

Combinations (Order Does Not Matter)

A **combination** is a selection of objects where order does not matter, and no repetitions are allowed.

1. Picking three players to form a team out of n candidates (without assigning positions).
2. Choosing two desserts from a tray: it does not matter which one you pick first or second.

We call these numbers **binomial coefficients** and denote them by

$$\binom{n}{k}.$$

Binomial coefficients answer several equivalent questions:

$$\begin{aligned}\binom{n}{k} &= \text{the number of } k\text{-element subsets of an } n\text{-element set,} \\ \binom{n}{k} &= \text{the number of binary strings of length } n \text{ with exactly } k \text{ ones.}\end{aligned}$$

Formula for Binomial Coefficients

It turns out that there is a simple formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)! k!}.$$

Compare this to the number of *permutations* of k objects chosen from n :

$$P(n, k) = {}_n P_k = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Example: There are

$$P(5, 2) = 5 \cdot 4 = 20$$

ways to choose two students out of 5 if the order matters, but only

$$\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$$

ways if the order does not matter.

Why Do We Divide by $k!$ in the Combinations Formula?

Let's begin by comparing **permutations** and **combinations**.

A *permutation* counts arrangements where order matters, while a *combination* counts selections where order does not matter. Because every group of k objects can be arranged in $k!$ different orders, there are always more permutations than combinations. In fact, each combination corresponds to $k!$ different permutations.

Consider this example. We have 7 students, and we want to award three medals:

Gold (G), Silver (S), and Bronze (B).

We can count the possibilities in two different ways.

- **Method 1: Using permutations.** First choose who gets Gold (7 choices), then Silver (6 choices), then Bronze (5 choices):

$$7 \times 6 \times 5 = 210.$$

- **Method 2: Using combinations.** Suppose we first choose the 3 medal winners, without caring who gets which medal. That number of selections is $\binom{7}{3}$. Then, once the 3 people are chosen, we can assign the medals in $3! = 6$ different orders (3 choices for Gold, 2 choices for Silver, 1 choice for Bronze):

$$\binom{7}{3} \times 3! = 210.$$

Solving for $\binom{7}{3}$ gives

$$\binom{7}{3} = \frac{210}{3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35.$$

So there are 35 different groups of medal winners when order doesn't matter.

Both methods describe the same situation, but Method 2 divides by $3!$ to remove overcounting caused by rearranging the same three people. This shows that to obtain combinations from permutations, we divide by $3!$ — the number of ways to reorder the same 3 chosen people.

Let's now look at this idea in general form. If we count using permutations, the number of ways to choose and order k people out of n is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

But each group of k people can appear in $k!$ different orders, all representing the same combination. Dividing by $k!$ corrects for this overcounting:

$$\boxed{\binom{n}{k} = \frac{n!}{(n-k)! k!}}.$$

So the combinations formula simply says:

$$\text{Combinations} = \frac{\text{Permutations}}{\text{Rearrangements of each group}}.$$

It's the same reasoning we saw in the medal example — each set of winners has $k!$ internal orders that we don't distinguish.

Alternative Explanation.

Let's start with a smaller case: choosing 2 people out of a group of n . If we count using permutations, the number of ways is

$$n \cdot (n-1).$$

This counts ordered pairs: first person, second person. But when we form a *committee of 2*, order does not matter. The pair (Alice, Bob) is the same as (Bob, Alice). Thus, every pair has been counted *twice*. To fix this, we divide by $2!$:

$$\binom{n}{2} = \frac{n(n-1)}{2!}.$$

Now imagine choosing 3 people. Counting ordered triples gives

$$n \cdot (n-1) \cdot (n-2).$$

But any group of 3 people can be arranged in $3!$ different orders. For example, the group {Alice, Bob, Carol} could appear as (Alice, Bob, Carol), (Alice, Carol, Bob), (Bob, Alice, Carol), and so on — in total, $3! = 6$ different permutations. All of them represent the same combination, so we must divide by $3!$:

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}.$$

This simple case illustrates the same principle: every unordered selection corresponds to multiple ordered arrangements, and dividing by $k!$ removes this overcounting.

Useful Identities for Combinations

1. Symmetry Identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

Choosing k elements from n is the same as choosing which $(n-k)$ elements to *leave out*.

Example. Choosing 2 students out of 5 is equivalent to choosing the 3 students who are *not* selected:

$$\binom{5}{2} = \binom{5}{3} = 10.$$

2. Pascal's Identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Imagine forming a group of k students from $n + 1$ total. Either a particular student (say, Alice) is included — then you must choose $(k - 1)$ of the remaining n — or she is not included — then you must choose all k from the remaining n . The total number of possible groups is the sum of those two cases.

Example. For $n = 4$, $k = 2$:

$$\binom{4}{2} + \binom{4}{1} = 6 + 4 = 10 = \binom{5}{2}.$$

Combinations with Repetition (Multisets)

Sometimes we choose objects where **order does not matter**, but we **allow repeats**. Such selections are called **combinations with repetition** or **multisets**.

Example 1. You are choosing 3 scoops of ice cream from 5 available flavors (vanilla, chocolate, strawberry, pistachio, mango). You may choose the same flavor more than once, e.g. “vanilla–vanilla–chocolate.” How many different bowls of 3 scoops are possible?

Let's look at two extreme ways to count and see why both are incorrect.

- **If we treat scoops as ordered:** For each of the 3 positions, we can choose any of the 5 flavors. That gives

$$5^3 = 125$$

possibilities. But this *overcounts*, because the order of scoops doesn't matter — a bowl with (vanilla, chocolate, vanilla) is the same as (vanilla, vanilla, chocolate).

- **If we treat scoops as all different:** Using ordinary combinations, we might try

$$\binom{5}{3} = 10,$$

which counts ways to pick 3 *distinct* flavors. But this *undercounts*, because it doesn't allow repeats like “chocolate–chocolate–mango.”

So neither 5^3 (which allows order) nor $\binom{5}{3}$ (which forbids repeats) fits the problem. We need a method that allows *repetition* but ignores *order*. That situation is called a **combination with repetition**.

Order doesn't matter (vanilla–chocolate–vanilla is the same as chocolate–vanilla–vanilla), but repetitions are allowed.

This situation is counted by the formula:

$$\text{Number of combinations with repetition} = \binom{n + k - 1}{k},$$

where

- n = number of types of objects (flavors, colors, etc.)
- k = number of items selected

Why Does the Formula Work?

We can represent each choice as a sequence of stars and bars:

- Each **star** (*) represents one item chosen.
- Each **bar** (|) separates categories (flavors, colors, etc.).

Example: choosing 3 scoops from 5 flavors.

$(** || * | |)$ means: 2 of flavor 1, 0 of flavor 2, 1 of flavor 3, 0 of flavors 4 and 5.

We have k stars and $(n - 1)$ bars. Altogether there are $k + n - 1$ positions, and we must choose which k of them are stars (the rest become bars):

$$\binom{n + k - 1}{k}.$$

Example (continued): For $n = 5$ flavors and $k = 3$ scoops:

$$\binom{5 + 3 - 1}{3} = \binom{7}{3} = 35.$$

There are 35 possible bowls of three scoops (allowing repeats).

Summary

Combinatorics at a Glance

Key Principles:

- **Multiplication Rule:** If two tasks can be done in m and n ways respectively, total = $m \times n$.
- **Factorial:** $n! = n \cdot (n - 1) \cdot \dots \cdot 1$, with $0! = 1$.

Concept	When Order Matters	When Order Does NOT Matter
Without Repetition	$P(n, k) = \frac{n!}{(n - k)!}$	$\binom{n}{k} = \frac{n!}{(n - k)! k!}$
With Repetition Allowed	n^k	$\binom{n + k - 1}{k}$

Useful Identities:

$$\binom{n}{k} = \binom{n}{n - k}, \quad \binom{n}{k} + \binom{n}{k - 1} = \binom{n + 1}{k}.$$

*Permutations = arrangements where order matters.
Combinations = selections where order doesn't matter.*

Sample Problems

1. **(Multiplication Rule)** A sandwich shop offers 4 breads, 3 proteins, and 5 toppings (choose exactly one of each). How many distinct sandwiches are possible?
2. **(Complement Counting)** A 4-digit code uses digits 0–9, and digits may repeat.
 - (a) How many codes are possible?
 - (b) How many codes contain at least one digit 7?
3. **(Seating With a Constraint)** Six students sit in a row. Two of them, Alice and Bob, insist on sitting next to each other. In how many ways can the six students be seated?
4. **(Order vs. No Order)** From 7 finalists, we must:
 - (a) Award gold, silver, and bronze medals (all different people).
 - (b) Choose a 3-person committee (no roles).Compute both numbers and briefly explain why they differ.
5. **(Teams With/Without a Role)** From a class of 12 students:
 - (a) How many 4-student teams can be formed?
 - (b) How many 4-student teams can be formed if one of the 4 must be designated captain?
6. **(Grid Paths)** You can only move *right* (R) or *up* (U) along grid lines.
 - (a) How many shortest paths are there from $(0, 0)$ to $(4, 3)$?
 - (b) How many such paths pass through the point $(2, 1)$?
7. **(Arrangements with Repeats)** How many distinct rearrangements of the letters in BANANA are there?
8. **(Combinations with Repetition)** A candy jar has 10 identical candies in total and 4 children (Amy, Ben, Chloe, Dan).
 - (a) In how many ways can the candies be distributed if a child may receive any number of candies (including zero)?
 - (b) In how many ways can the candies be distributed if each child must receive at least one candy?
9. **(Non-Attacking Rooks)** On a 6×6 chessboard, in how many ways can we place 4 rooks so that no two attack each other? (Hint: Choose rows, choose columns, then assign.)

Solutions to Sample Problems

1. (**Multiplication Rule**) A sandwich shop offers 4 breads, 3 proteins, and 5 toppings (choose exactly one of each). How many distinct sandwiches are possible?

Solution. Choices are independent: pick 1 bread (4 ways), 1 protein (3 ways), 1 topping (5 ways). By the multiplication rule,

$$4 \cdot 3 \cdot 5 = 60.$$

2. (**Complement Counting**) A 4-digit code uses digits 0–9, and digits may repeat.

- (a) How many codes are possible?
- (b) How many codes contain at least one digit 7?

Solution.

- (a) Each of 4 positions has 10 choices (0–9), so $10^4 = 10000$.
- (b) Let A = “code has at least one 7.” Then complement to A A^c = “no 7 anywhere,” so each position has 9 choices (excluding 7): 9^4 codes. Thus

$$|A| = 10^4 - 9^4 = 10000 - 6561 = 3439.$$

3. (**Seating With a Constraint**) Six students sit in a row. Two of them, Alice and Bob, insist on sitting next to each other. In how many ways can the six students be seated?

Solution. Treat $[AB]$ as a single block (they must be adjacent). Then we have 5 items to arrange: the block plus the other 4 students. That gives $5!$ linear orders for the blocks, and inside the block we can have AB or BA (2 ways). Hence

$$5! \cdot 2 = 120 \cdot 2 = 240.$$

4. (**Order vs. No Order**) From 7 finalists, we must:

- (a) Award gold, silver, and bronze medals (all different people).
- (b) Choose a 3-person committee (no roles).

Compute both numbers and briefly explain why they differ.

Solution.

- (a) Ordered awards (roles matter): permutations ${}_7P_3 = 7 \cdot 6 \cdot 5 = 210$.
- (b) Unordered committee: combinations $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$.

They differ because in (a) order/roles matter; in (b) order does not. In fact,

$$7 \cdot 6 \cdot 5 = \binom{7}{3} \cdot 3!,$$

i.e., permutations = combinations $\times$ internal reorderings.

5. (**Teams With/Without a Role**) From a class of 12 students:

- (a) How many 4-student teams can be formed?
- (b) How many 4-student teams can be formed if one of the 4 must be designated captain?

Solution.

- (a) Unordered team: $\binom{12}{4} = 495$.

- (b) Choose the team and then choose the captain among the 4:

$$\binom{12}{4} \cdot 4 = 495 \cdot 4 = 1980.$$

(Equivalently, choose captain first then 3 others: $12 \cdot \binom{11}{3} = 12 \cdot 165 = 1980$.)

6. (**Grid Paths**) You can only move *right* (R) or *up* (U) along grid lines.

- (a) How many shortest paths are there from $(0, 0)$ to $(4, 3)$?
 (b) How many such paths pass through the point $(2, 1)$?

Solution.

- (a) A shortest path uses exactly 4 R's and 3 U's in some order: total 7 steps. Choose the positions of, say, the R's:

$$\binom{7}{4} = \binom{7}{3} = 35.$$

- (b) Count paths $(0, 0) \rightarrow (2, 1)$ times paths $(2, 1) \rightarrow (4, 3)$.

$$(0, 0) \rightarrow (2, 1) : \text{needs 2 R and 1 U} \Rightarrow \binom{3}{2} = 3. \quad (2, 1) \rightarrow (4, 3) : \text{needs 2 R and 2 U} \Rightarrow \binom{4}{2} = 6.$$

Multiply:

$$3 \cdot 6 = 18.$$

7. (**Arrangements with Repeats**) How many distinct rearrangements of the letters in BANANA are there?

Solution. There are 6 letters total with repeats: *A* appears 3 times, *N* appears 2 times, *B* appears once. Number of distinct permutations of a multiset:

$$\frac{6!}{3!2!} = \frac{720}{6 \cdot 2} = 60.$$

8. (**Combinations with Repetition**) A candy jar has 10 identical candies in total and 4 children (Amy, Ben, Chloe, Dan).

- (a) In how many ways can the candies be distributed if a child may receive any number of candies (including zero)?
 (b) In how many ways can the candies be distributed if each child must receive at least one candy?

Solution. Let x_A, x_B, x_C, x_D be the numbers of candies received by Amy, Ben, Chloe, and Dan.

- (a) We need the number of *nonnegative integer* solutions to

$$x_A + x_B + x_C + x_D = 10, \quad x_A, x_B, x_C, x_D \geq 0.$$

Think of the 10 candies as stars (*) and use 3 bars to split them into 4 groups (one per child). For example, ** | **** | | *** means $(2, 4, 0, 3)$. There are 10 stars and 3 bars arranged in a line: total positions $10 + 3 = 13$, choose where the bars go:

$$\binom{10 + 4 - 1}{4 - 1} = \binom{13}{3} = 286.$$

- (b) Now require each child to receive at least one candy, i.e.,

$$x_A + x_B + x_C + x_D = 10, \quad x_A, x_B, x_C, x_D \geq 1.$$

Give each child 1 candy first (to satisfy the “at least one” condition). That uses 4 candies, leaving $10 - 4 = 6$ to distribute freely with zeros allowed:

$$y_A + y_B + y_C + y_D = 6, \quad y_i \geq 0.$$

By the same stars-and-bars reasoning,

$$\binom{6+4-1}{4-1} = \binom{9}{3} = 84.$$

9. **(Non-Attacking Rooks)** On a 6×6 chessboard, in how many ways can we place 4 rooks so that no two attack each other? (Hint: Choose rows, choose columns, then assign.)

Solution. Choose the 4 rows: $\binom{6}{4}$; choose the 4 columns: $\binom{6}{4}$; then place one rook in each chosen row/column (4 intersections in the first column, 3 remaining intersections in the second column, etc): $4!$ ways. Thus

$$\binom{6}{4} \binom{6}{4} \cdot 4! = 15 \cdot 15 \cdot 24 = 5400.$$

Homework

Note: Unless a problem *explicitly asks for a numeric answer*, you may leave your result in terms of factorials or binomial coefficients (for example, $\binom{10}{3}$ or $\frac{10!}{7!3!}$). Simplifying to a number is not required.

1. A dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses, and 3 desserts. How many possible dinners are there?
2. (a) A soccer club has 15 students. The coach must select 11 players to form the team for a match. How many possible teams can be chosen?
(b) The same soccer club has 15 students. This time the coach must select 10 players plus 1 goal-keeper. How many possible teams are there now? (*Hint: In part (a) only the set of 11 matters; in part (b) one player must also be designated as goalkeeper.*)
3. A chess club has 2 girls and 7 boys. A team of four players must be chosen for a tournament, and the team must include at least one girl. In how many ways can this be done?
4. In how many ways can 10 people be divided into two basketball teams of 5 players each?
5. In a certain language, there are 3 vowels and 5 consonants. A syllable consists of any vowel and any consonant in either order, and a word consists of any two syllables. How many different words exist in this language?
6. In how many ways can 8 different candies be distributed into 2 different bags, if
 - a) all candies must be distributed and the bags may be empty;
 - b) all candies must be distributed and the bags cannot be empty;
 - c) not all candies must be distributed and the bags may be empty?
7. A person runs down a staircase of 12 steps (including the top and bottom). He may jump over some steps.
 - (a) In how many ways can he reach the bottom in exactly 5 jumps?
 - (b) In how many ways can he reach the bottom with no restriction on the number of jumps? (*Hint: keep track of which steps he actually lands on.*)
8. Alex the Trickster has ten fake IDs. To stay undercover, each time he meets a police officer, he shows a different ID than the one he showed to the previous officer and also different from the one he showed to the officer before that. In how many different ways can Alex show his IDs to ten officers?
9. How many ways are there to divide 12 books:
 - (a) between two bags,
 - (b) between two bookshelves (order on each shelf matters),
 - (c) between three bags,
 - (d) between three bookshelves (order on each shelf matters)?
10. A store sells $n = 4$ types of balloons: red, blue, green, yellow. You need to buy $k = 6$ balloons in total, and colors may repeat. How many possible color combinations are there?
11. A holiday light garland hanging along a school hallway consists of red and blue bulbs. Each red bulb must have at least one blue bulb next to it. What is the maximum possible number of red bulbs if the total number of bulbs is 50?
- *12. Eight spotlights are arranged in a row at equal distances, all facing the sea. From an approaching ship, one can distinguish only the relative positions of the lit spotlights (their order and spacing), but not their absolute position relative to the shore.

For example, if only one spotlight is on, the observer cannot tell which one it is.

How many different signals can be sent to the ship using these spotlights if at least one spotlight must be on? Give a numerical answer.

- *13. In the town of Springfield, every family used to live in a separate house. One day, each family moved into a house that was previously occupied by another family. It was decided to repaint all the houses in red, blue, or green so that for every family, the color of their new house is different from the color of their old one. Is it possible to do this?
- *14. A row of 100 children of different heights is standing in line. You are allowed to choose any 50 consecutive children and rearrange them among themselves in any order (the others remain in place). How can you, in only six such rearrangements, guarantee that all the children will end up ordered by decreasing height from left to right?