MATH 7: HANDOUT 7

GEOMETRIC SEQUENCES

Geometric Sequences

Definition

Arithmetic sequences grow by adding the same number each step. **Geometric sequences**, in contrast, grow by multiplying by the same number each step.

This simple idea shows up everywhere:

- Population growth and compound interest (exponential growth).
- Radioactive decay (exponential decay).
- Physics of vibrations and waves.
- Computer science (algorithms that double or halve data sizes).
- Music (halftones in the chromatic scale).

A **geometric sequence** (or geometric progression) is a sequence in which each term is obtained from the previous one by multiplying by a fixed number, called the **common ratio** q.

Example: The sequence

$$6,\ 12,\ 24,\ 48,\ \ldots$$

is geometric with $a_1 = 6$ and $q = 2$.

General Term

The n-th term of a geometric sequence is

$$a_n = a_1 q^{n-1}.$$

Example: If $a_1 = 6$ and $q = 2$, then

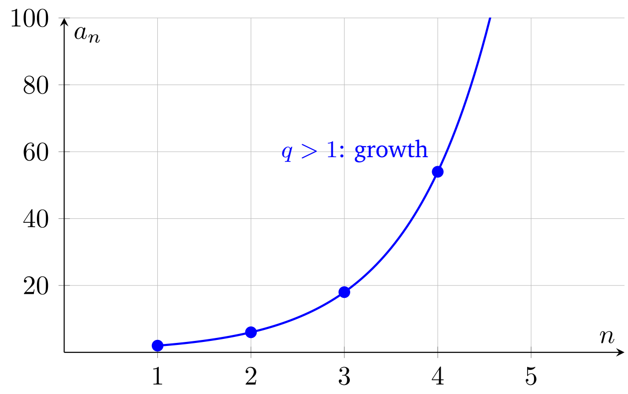
$$a_{10} = 6 \cdot 2^9 = 6 \cdot 512 = 3072.$$

Behavior of the Terms of the Geometric Sequence

If we graph the terms of a geometric sequence, the points no longer lie on a straight line as they did in case of arithmetic sequence — they form a curve that reflects how the values grow or shrink depending on q.

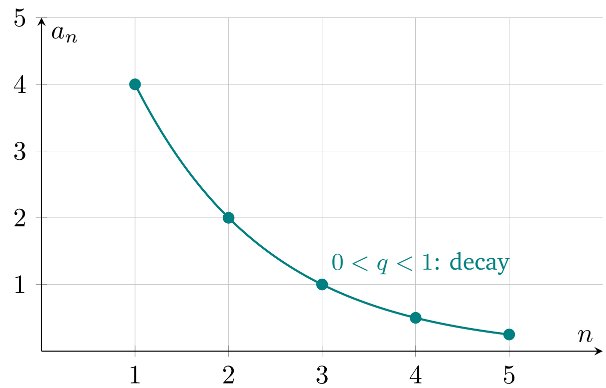
Case 1: $q > 1$ (Exponential Growth)

When $q > 1$, each term is larger than the previous one. The sequence grows rapidly — this is called **exponential growth**.



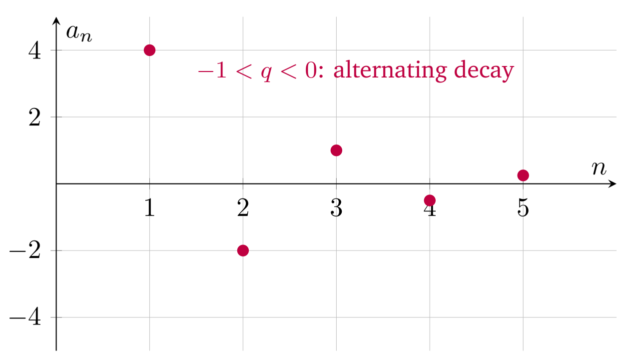
Case 2: $0 < q < 1$ (Exponential Decay)

When q is between 0 and 1, each term is smaller than the previous one. The sequence decreases but always stays positive — this is called **exponential decay**.



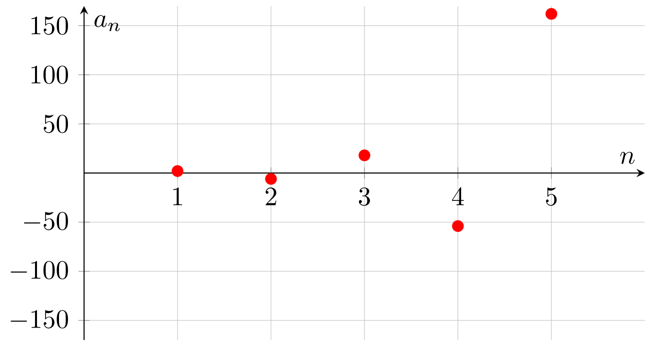
Case 3: $-1 < q < 0$ (Alternating Decay)

When q is negative but greater than -1, the terms alternate in sign (positive, negative, positive, . . .), while their absolute values get smaller and smaller. The graph "bounces" above and below the axis, approaching zero.



Case 4: $q < -1$ (Alternating Growth)

When q is less than -1, the terms alternate in sign and grow in absolute value. The graph still "bounces," but now the values get larger and larger.



Behavior of Geometric Sequences

- If $q > 1$, the sequence shows **exponential growth** — terms increase rapidly.
- If $0 < q < 1$, the sequence shows **exponential decay** — terms decrease toward 0.
- If $-1 < q < 0$, terms alternate in sign and their magnitudes shrink toward 0.
- If $q < -1$, terms alternate in sign and grow in magnitude.

Geometric Mean Property

In a geometric sequence, each term is the **geometric mean** of its neighbors:

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}.$$

Proof: Since $a_n = a_{n-1}q$ and $a_{n+1} = a_n q$, multiplying gives

$$a_{n-1} \cdot a_{n+1} = (a_n/q) \cdot (a_n q) = a_n^2.$$

■

Sum of n Terms

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1 - q^n)}{1 - q}$$

Proof: To prove this, we write the sum and multiply it by q:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= qa_1 + qa_2 + \cdots + qa_n \end{aligned}$$

Now notice that $qa_1 = a_2, \ldots qa_2 = a_3, \ldots, qa_n = a_{n+1}$, etc, so we have:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= a_2 + a_3 + \cdots + a_{n+1} \end{aligned}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$\begin{aligned} S_n - qS_n &= (a_1 - a_{n+1}), \text{ or} \\ S_n(1 - q) &= (a_1 - a_1 q^n) \\ S_n(1 - q) &= a_1(1 - q^n) \end{aligned}$$

from which we get the formula above. ■

Example Find $1 + 2 + 4 + 8 + \cdots + 2^{10}$. Here $a_1 = 1$, $q = 2$, $n = 11$.

$$S_{11} = \frac{1(1 - 2^{11})}{1 - 2} = 2^{11} - 1 = 2047.$$

Sum and Product Notation

Mathematicians use special symbols to write long sums or products compactly.

1. Sum notation (Sigma notation).

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

The Greek letter Σ (sigma) means "sum of." For example,

$$\begin{aligned}\sum_{k=1}^{5} 2^k &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 2 + 4 + 8 + 16 + 32 = 62.\end{aligned}$$

2. Product notation (Pi notation).

$$\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n$$

The Greek letter Π (pi) means "product of." For example,

$$\prod_{k=1}^{4} k = 1 \cdot 2 \cdot 3 \cdot 4 = 4! = 24.$$

3. Application to geometric sequences.
A geometric sum can be written neatly using Σ:

$$S_n = \sum_{k=1}^{n} a_1 q^{k-1}.$$

The k-th term is $a_k = a_1 q^{k-1}$, and the sum automatically includes all terms from a_1 to a_n.

These notations make formulas shorter and easier to read, especially for large n.

Infinite Sum

If $0 < q < 1$, then as $n \to \infty$, the terms $a_1 q^n$ get smaller and smaller, approaching 0. So the sum approaches a limit:

$$S = \frac{a_1}{1 - q}.$$

Proof The sum of the first n terms is

$$S_n = a_1 + a_1 q + a_1 q^2 + \cdots + a_1 q^{n-1} = a_1 \frac{1 - q^n}{1 - q}.$$

When $|q| < 1$, powers of q become very small: $q^n \to 0$.

Therefore, as $n \to \infty$,

$$S_n = a_1 \frac{1-q^n}{1-q} \to a_1 \frac{1-0}{1-q} = \boxed{\frac{a_1}{1-q}}.$$

■

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

This idea underlies many real-world models:

- A bouncing ball loses half its height each bounce.
- A discount factor in finance.
- An infinite repeating decimal, e.g. $0.\overline{9} = 1$.

  Let us look at this example in a bit more details. Why do we assume that $0.999\ldots = 1$? Indeed,

$$\begin{aligned} 0.9999\ldots &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots \\ &= \frac{9}{10} + \frac{9}{10} \cdot \left(\frac{1}{10}\right) + \frac{9}{10} \cdot \left(\frac{1}{10}\right)^2 + \cdots. \end{aligned}$$

  We can observe that this is a sum of geometric sequence with $a_1 = \frac{9}{10}$ and $q = \frac{1}{10}$. Using the formula for infinite geometric sequence, we get:

$$0.999\ldots = \frac{a_1}{1-q} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1.$$

Depreciation of Cars/Phones

Some items lose the same $\%$ of value each year:

$$V_n = V_1 \cdot (1-p)^{n-1},$$

e.g., a phone losing 20% yearly: $V_n = V_1 \cdot 0.8^{\,n-1}$.

Population Growth/Decay

If a population changes by a fixed $\%$ per step:

$$P_n = P_1 \cdot (1+r)^{n-1} \quad (r > 0 \text{ growth},\ r < 0 \text{ decay}).$$

Example: bacteria doubling each hour: $P_n = P_1 \cdot 2^{n-1}$.

Bouncing Ball Heights

A ball rebounds to a fixed fraction q of its previous height:

$$h_n = h_1 \cdot q^{\,n-1}.$$

If $q = \frac{3}{5}$, heights go $h_1,\ \frac{3}{5}h_1,\ \left(\frac{3}{5}\right)^2 h_1, \ldots$. Total vertical distance uses a geometric sum.

Compound Interest & Savings Growth

Money that earns a fixed percentage each period grows by a geometric sequence:

$$a_n = a_1 \cdot (1+r)^{n-1}.$$

Example: \$1000 at $r = 5\%$ per year becomes $1000,\ 1050,\ 1102.5, \cdots = 1000 \cdot 1.05^{n-1}$.
Setup. You deposit the same amount d at the *end* of each period (month, year, etc.). The account pays a fixed interest rate r *per period*. After n deposits, how much money is in the account?

Idea. Each deposit grows for a different number of periods. Let $q = 1 + r$. The last deposit grows for 0 periods, the one before grows for 1 period, etc.

$$S_n = d\big(q^{n-1} + q^{n-2} + \cdots + q^1 + q^0\big) = d \cdot \frac{q^n - 1}{q - 1}.$$

Future value (end-of-period deposits):

$$\boxed{S_n = d \cdot \frac{(1+r)^n - 1}{r}}$$ (also called an annuity-immediate).

If deposits are at the *beginning* of each period (each payment earns one extra period of interest), multiply by q:

$$\boxed{S_n^{(\text{begin})} = (1+r)\, d \cdot \frac{(1+r)^n - 1}{r}}$$ (annuity-due).

Example (end-of-month deposits). Deposit $d = \$200$ each month at $r = 0.5\%\ (= 0.005)$ per month for $n = 12$ months:

$$S_{12} = 200 \cdot \frac{(1.005)^{12} - 1}{0.005} \approx 200 \cdot 12.34 \approx \$2{,}468.$$

Without interest, you'd have \$2,400; interest adds about \$68.

Medicine in the Body

If the body removes a fixed $\%$ of medicine each hour, the amount left after each hour forms

$$M_n = M_1 \cdot (1-p)^{n-1}.$$

This describes **exponential decay**: the medicine concentration decreases by the same fraction every hour rather than by a fixed amount.

Example. If 20% of a drug is eliminated each hour ($p = 0.20$), then after 3 hours only

$$(1 - 0.20)^3 = 0.8^3 = 0.512$$

or about 51% of the original dose remains.

When a new dose is taken before the previous one is (almost) fully gone, the remaining amounts add up, forming a **geometric series**. This is why doctors set dosing intervals carefully—to keep the concentration within a safe and effective range (the *therapeutic window*) without the medicine building up to toxic levels.

Radioactive Half-Life and Carbon Dating

A substance that halves every k units of time follows

$$A_n = A_0 \cdot \left(\tfrac{1}{2}\right)^n.$$

Each "half-life" step multiplies by $q = \frac{1}{2}$.

Carbon Dating. Carbon-14 is a radioactive isotope of carbon that is constantly produced in the upper atmosphere by cosmic rays. Living things absorb carbon—both the stable ^{12}C and the radioactive ^{14}C—through breathing and eating. As long as an organism is alive, the ratio of ^{14}C to ^{12}C in its body stays nearly the same as in the atmosphere.

When the organism dies, it stops exchanging carbon with the environment. From that moment, the ^{14}C inside begins to decay while ^{12}C remains stable. By measuring how much ^{14}C is left in a fossil and comparing it to the known ratio in modern living organisms, scientists can determine what fraction of carbon-14 remains—and thus how long it has been decaying.

Carbon-14 has a half-life of about 5730 years. If a fossil contains only 25% of the original carbon-14, that means two half-lives have passed:

$$\left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4},$$

so the fossil is roughly $2 \times 5730 = 11{,}460$ years old.

Tournaments and Brackets

In single-elimination play, the number of teams each round follows

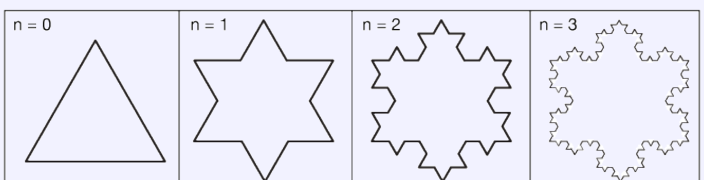
$$N,\ \tfrac{N}{2},\ \tfrac{N}{4}, \ldots$$

– a geometric sequence with $q = \frac{1}{2}$.

Fractals (e.g., Koch Snowflake)

At each stage, the number and length of segments change by fixed factors, so the perimeter terms follow a **geometric progression**, while the total area approaches a finite limit that can be found by summing a geometric series.

Fractals like the Koch Snowflake show how geometric sequences appear in nature and art— finite area, yet infinite perimeter!



Number of sides (N)	3	12	48	192
Side length (S)	1	1/3	1/9	1/27
Perimeter length (P)	3	4	5.33	9.11

Koch Snowflake construction

Common Mistakes

- Confusing $a_n = a_1 q^n$ with the correct $a_n = a_1 q^{n-1}$.
- Forgetting the sum formula only works for $q \neq 1$.
- Using the infinite sum formula when $|q| \geq 1$ (it only converges if $|q| < 1$).

Sample Problems

1. **Geometric or not? Find q and the next three terms (if geometric).**
   (a) $3,\ 6,\ 12,\ 24,\ \dots$
   (b) $-5,\ 10,\ -20,\ 40,\ \dots$
   (c) $2,\ 5,\ 12.5,\ 31.25,\ \dots$
   (d) $7,\ 7,\ 7,\ 7,\ \dots$
   (e) $4,\ 8,\ 13,\ 21,\ \dots$
2. **General term and specific values.**
   (a) $a_1 = 9,\ q = \frac{1}{3}$. Find a_n and a_7.
   (b) $a_4 = 48,\ q = 2$. Find a_1 and a_8.
   (c) $a_1 = -3,\ a_5 = 48$. Find q and a_{10}.
3. **Geometric mean property.** In a geometric sequence (a_n), $a_3 = 12$ and $a_5 = 48$.
   (a) Find a_4 using $a_4 = \sqrt{a_3 a_5}$.
   (b) Find the common ratio q and a_1.
4. **Finite sums.**
   (a) Compute $S = \ 5 + 10 + 20 + 40 + \cdots + 5 \cdot 2^9$.
   (b) Compute $T = \ 1 - 4 + 4^2 - 4^3 + \cdots + (-4)^{10}$.
   (c) Compute $U = \ \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^{12}}$.
5. **Infinite geometric series.** For each series, state whether it converges. If it converges, find the sum.
   (a) $1 + \dfrac{1}{4} + \dfrac{1}{4^2} + \dfrac{1}{4^3} + \dots$
   (b) $3 - \dfrac{3}{5} + \dfrac{3}{5^2} - \dfrac{3}{5^3} + \dots$
6. **Savings with monthly deposits (compound interest).** You deposit $d = \$150$ at the *end* of each month into an account earning 0.6% per month.
   (a) Find the amount after $n = 12$ months.
   (b) How much is in the account after $n = 60$ months (5 years)?
   (c) If you instead deposit at the *beginning* of each month, how do your answers change?
   (d) Compare your results with the situation when there is *no interest at all*. How much of the total balance is due to your deposits and how much comes from interest?
7. **Bouncing ball total distance.** A ball is dropped from height H meters. Each bounce reaches $\frac{3}{5}$ of the previous height.
   (a) Write the sequence of rebound heights.
   (b) Find the total vertical distance traveled by the ball until it stops bouncing (assume infinitely many bounces).
   (c) If $H = 2$ meters, compute the total distance numerically.

Sample Problems: Solutions

1. **Geometric or not? Find q and the next three terms (if geometric).**

   (a) $3, 6, 12, 24, \ldots$ is geometric with $q = 2$. Next: $48, 96, 192$.

   (b) $-5, 10, -20, 40, \ldots$ is geometric with $q = -2$. Next: $-80, 160, -320$.

   (c) $2, 5, 12.5, 31.25, \ldots$ is geometric with $q = \frac{5}{2}$. Next: $78.125,\ 195.3125,\ 488.28125$.

   (d) $7, 7, 7, 7, \ldots$ is geometric with $q = 1$. Next: $7, 7, 7$.

   (e) $4, 8, 13, 21, \ldots$ is *not* geometric (ratios are not constant).

2. **General term and specific values.**

   (a) $a_1 = 9,\ q = \frac{1}{3}$. Then $a_n = 9\left(\frac{1}{3}\right)^{n-1}$ and

   $$a_7 = 9\left(\tfrac{1}{3}\right)^6 = \frac{9}{729} = \frac{1}{81}.$$

   (b) $a_4 = 48,\ q = 2$. Since $a_4 = a_1 q^3$, we get $a_1 = \frac{48}{2^3} = 6$. Then

   $$a_8 = a_1 q^7 = 6 \cdot 2^7 = 6 \cdot 128 = 768.$$

   (c) $a_1 = -3,\ a_5 = 48$. Here $a_5 = a_1 q^4 \Rightarrow q^4 = \frac{48}{-3} = -16$, which has no real solution. *Conclusion: no real geometric sequence fits these data.*

3. **Geometric mean property.** Given $a_3 = 12,\ a_5 = 48$.

   (a) Since $a_n^2 = a_{n-1}a_{n+1}$, we have $a_4^2 = 12 \cdot 48 = 576$, so $a_4 = \pm 24$.

   (b) Also $a_5 = a_3 q^2 \Rightarrow q^2 = \frac{48}{12} = 4$, so $q = 2$ or $q = -2$. Then

   $$a_1 = \frac{a_3}{q^2} = \frac{12}{4} = 3.$$

   If $q = 2$ then $a_4 = 12 \cdot 2 = 24$; if $q = -2$ then $a_4 = 12 \cdot (-2) = -24$.

4. **Finite sums.**

   (a) $S = 5 + 10 + 20 + \cdots + 5 \cdot 2^9$ has $a_1 = 5,\ q = 2,\ n = 10$:

   $$S = 5 \cdot \frac{2^{10} - 1}{2 - 1} = 5(1024 - 1) = 5 \cdot 1023 = 5115.$$

   (b) $T = 1 - 4 + 4^2 - 4^3 + \cdots + (-4)^{10} = \sum_{k=0}^{10} (-4)^k$:

   $$T = \frac{1 - (-4)^{11}}{1 - (-4)} = \frac{1 - (-4194304)}{5} = \frac{4194305}{5} = 838861.$$

   (c) $U = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{12}}$ has $a_1 = \frac{1}{2},\ q = \frac{1}{2},\ n = 12$:

   $$U = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{12}\right)}{1 - \frac{1}{2}} = 1 - \frac{1}{4096} = \frac{4095}{4096}.$$

5. **Infinite geometric series.** Converges iff $|q| < 1$; sum $= \dfrac{a_1}{1 - q}$.

(a) $1+\dfrac{1}{4}+\dfrac{1}{4^2}+\dots$: $a_1=1,\ q=\frac{1}{4}$. Converges to $\dfrac{1}{1-\frac{1}{4}}=\dfrac{4}{3}$.

(b) $3-\dfrac{3}{5}+\dfrac{3}{5^2}-\dots$: $a_1=3,\ q=-\frac{1}{5}$. Converges to $\dfrac{3}{1-(-\frac{1}{5})}=\dfrac{3}{\frac{6}{5}}=\dfrac{5}{2}$.

6. **Savings with monthly deposits (compound interest).** With end-of-month deposits,

$$S_n = d\cdot\frac{(1+r)^n-1}{r},\qquad d=150,\ r=0.006.$$

(a) $n=12$:

$$S_{12}=150\cdot\frac{(1.006)^{12}-1}{0.006}\approx 150\cdot 12.4040\approx \boxed{\$1{,}860.60}.$$

(b) $n=60$:

$$S_{60}=150\cdot\frac{(1.006)^{60}-1}{0.006}\approx \boxed{\$10{,}794.71}.$$

(c) Beginning-of-month deposits (annuity-due) multiply by $(1+r)$:

$$S_{12}^{(\text{begin})}\approx 1.006\cdot 1860.60\approx\boxed{\$1{,}871.77},\qquad S_{60}^{(\text{begin})}\approx 1.006\cdot 10794.71\approx\boxed{\$10{,}859.48}.$$

(d) Without interest, the total after n months is simply

$$S_{\text{no interest}}=d\cdot n.$$

- For $n=12$:

$$S_{\text{no interest}}=150\times 12=1800.$$

With interest, you had about \$1860.60, so the interest earned is roughly

$$1860.60-1800=\boxed{\$60.60}.$$

- For $n=60$:

$$S_{\text{no interest}}=150\times 60=9000.$$

With interest, you had about \$10,794.71, so the total interest earned is

$$10{,}794.71-9000=\boxed{\$1{,}794.71}.$$

7. **Bouncing ball total distance.** Rebound heights:

$$H,\ \tfrac{3}{5}H,\ \left(\tfrac{3}{5}\right)^2H,\ \dots$$

Total vertical distance

$$D=H+2\sum_{k=1}^{\infty}H\left(\tfrac{3}{5}\right)^k=H+2H\cdot\frac{\frac{3}{5}}{1-\frac{3}{5}}=H\cdot\frac{1+\frac{3}{5}}{1-\frac{3}{5}}=H\cdot\frac{\frac{8}{5}}{\frac{2}{5}}=4H.$$

For $H=2\,\text{m}$, $D=4\cdot 2=\boxed{8\text{ m}}$.

Homework

1. Write the first five terms of a geometric sequence if $a_1 = -20$ and $q = \frac{1}{2}$.
2. Determine the first two terms of the sequence
$$a_1,\ a_2,\ 24,\ 36,\ 54,\ \ldots$$
3. Find the common ratio of the geometric sequence
$$\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ \ldots$$
and compute a_{10}.
4. A geometric sequence has 99 terms. If $a_1 = 12$ and $a_{99} = 48$, find the 50th term.
5. Calculate:
$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{10}}.$$
6. Calculate:
$$1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots - 2^{15}.$$
7. Calculate:
$$1 - 3 + 3^2 - 3^3 + 3^4 - 3^5 + \cdots + 3^{20}$$
8. Calculate:
$$1 + x + x^2 + x^3 + \cdots + x^{100}.$$
9. A chessboard has 64 squares. If we place one grain of wheat on the first square, two on the second, four on the third, etc., approximately how many grains of wheat will there be in total? *Hint: use* $2^{10} \approx 10^3$.

   Estimate the total volume of this wheat and compare with the annual U.S. wheat harvest (about 2 billion bushels). *Data: one grain* ≈ 40 *mm*^{3}*, one bushel* ≈ 0.035 *m*^{3}.
10. Calculate an infinite sum:
$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots.$$
11. Calculate an infinite sum:
$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \ldots.$$
12. **Music and Frequency.** The connection between mathematics and music has been known since ancient times. The Greek philosopher Pythagoras (6th century BC) studied vibrating strings and discovered that musical harmony is based on numerical ratios. For example, halving the length of a string doubles the frequency, producing a higher note called an **octave**. Multiplying the frequency by $\frac{3}{2}$ gives the **perfect fifth**, another harmonious interval.

    Later, musicians developed a system of **equal temperament**, in which the octave is divided evenly into 12 equal parts called **semitones**. In this system, moving up by one semitone always multiplies the frequency by the same ratio q. After 12 semitones, the frequency doubles:
$$a_{13} = a_1 \cdot q^{12} = 2a_1.$$
    Today, this is the basis of Western music: the **chromatic scale**.

    If the note A has frequency 440 Hz, then the sequence of notes is
$$\ldots,\ A,\ A^\sharp,\ B,\ C,\ C^\sharp,\ D,\ D^\sharp,\ E,\ F,\ F^\sharp,\ G,\ G^\sharp,\ A,\ \ldots$$

(a) Find the value of the common ratio q.

(b) Compute the frequencies of the notes C and E above A.

(c) Using the calculator, find the ratio of frequencies of A and E (such an interval is called a **fifth**). Compare the frequency ratio with $\frac{3}{2}$ from Pythagoras' system. What do you notice?

Non-Western Traditions

Not all musical traditions use the 12-tone equal-tempered system. Many non-Western systems divide the octave differently or use continuous pitch changes rather than fixed steps. For example:

- **Indian classical music** uses a system of 22 microtones called *shrutis*.
- **Arabic and Turkish music** employ 24 quarter-tones per octave, allowing smaller melodic intervals.
- **Traditional Chinese music** often uses a *pentatonic* (five-note) scale based on pure frequency ratios derived from simple fractions like $3{:}2$.
- **Indonesian gamelan** tuning systems, such as *sléndro* (five tones) and *pélog* (seven tones), divide the octave into unequal steps unique to each ensemble.

Thus, while equal temperament is the foundation of modern Western instruments (like the piano), many world musical traditions use their own tunings that better suit the expressive and cultural character of their music.