MATH 7: HANDOUT 6

ARITHMETIC SEQUENCES

Introduction: What Are Sequences?

A **sequence** is simply an ordered list of numbers:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots$$

Each number in the list is called a **term** of the sequence, and each term has its position (first, second, third, and so on).

Some sequences are famous and extremely useful: the sequence of **natural numbers**

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$

the sequence of **odd numbers**

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots,$$

the sequence of **square numbers**

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots,$$

and the **Fibonacci sequence** (where each term equals the sum of the two previous ones)

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots$$

Each of these has a clear rule or pattern that allows us to find any term, even far ahead in the list.

Not all sequences are particularly **useful**. For example, consider the sequence

$$7,\ 13,\ 2,\ 19,\ 42,\ \ldots$$

or let a_n be the number of letters in the English word for n:

$a_1 = 3$ ("one"), $a_2 = 3$ ("two"), $a_3 = 5$ ("three"), $a_4 = 4$ ("four"), $a_5 = 4$ ("five"), $a_6 = 3$ ("six"), $\ldots$

This sequence depends entirely on the quirks of language rather than on mathematical structure. It is well defined, but not especially meaningful or useful in mathematics.

Some sequences, on the other hand, may **look simple but hide complicated rules**. Consider:

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$

It looks like the sequence of natural numbers ($a_n = n$), right? But it could also be defined by this bizarre formula:

$$a_n = (n-1)(n-2)(n-3)(n-4)(n-5) + n.$$

For $n = 1, 2, 3, 4, 5$, both formulas give the same values. But when $n = 6$, the result becomes very different: $a_6 = 126$.

This shows that just knowing the first few terms is not always enough to describe a sequence — we must also know the **rule** that generates them.

In this handout, we will focus on one of the simplest and most important types of sequences: the **arithmetic sequence**, where each term increases (or decreases) by the same constant amount.

Arithmetic Sequences

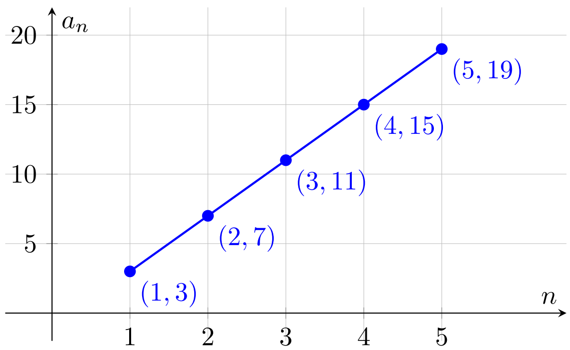
Definition

A sequence is called an **arithmetic sequence** (or arithmetic progression) if each term after the first is obtained by adding a fixed number, called the **common difference** d.

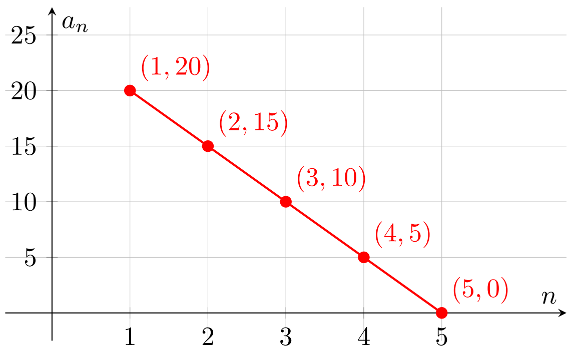
Example $3,\ 7,\ 11,\ 15,\ \ldots$ is an arithmetic sequence with $a_1 = 3$ and $d = 4$.

Each term is obtained by adding the same number, d, to the previous one. This means the sequence grows at a constant rate. Such growth is called **linear growth**: every step increases by the same amount.

If we graph the terms of this arithmetic sequence, plotting the term number n on the horizontal axis and the term value a_n on the vertical axis, all points lie on a single straight line. The line's slope is equal to the common difference d, and the vertical intercept corresponds to $a_1 - d$ (since $a_n = a_1 + (n-1)d$).



If d is **positive**, the sequence increases and the graph slopes upward — just like in the example above. If d is **negative**, the sequence decreases and the graph slopes downward. For instance, the sequence $20,\ 15,\ 10,\ 5,\ \ldots$ has $d = -5$, so each term is smaller than the previous one, and its graph is a straight line that goes down as n increases.



In both cases, all terms still lie on a straight line — that's what makes an arithmetic sequence **linear**.

Linear Growth vs Linear Decay

- If $d > 0$, the sequence shows **linear growth** — each term is larger than the previous one, and the graph slopes upward.
- If $d < 0$, the sequence shows **linear decay** — each term is smaller than the previous one, and the graph slopes downward.
- In both cases, the graph is a straight line because the change between terms is constant.

The n-th Term Formula

If a_1 is the first term and d is the common difference, then the n-th term is

$$a_n = a_1 + (n-1)d.$$

Examples

1. In the sequence $5,\ 8,\ 11,\ \ldots$, we have $a_1 = 5$, $d = 3$. The 20th term is

$$a_{20} = 5 + (20 - 1) \times 3 = 5 + 57 = 62.$$

2. Find the first term if $a_{15} = 50$ and $d = 3$:

$$a_1 = a_{15} - (15 - 1)d = 50 - 14 \times 3 = 8.$$

3. Find d if $a_1 = 12$ and $a_{10} = 39$:

$$\begin{aligned} a_{10} &= a_1 + (10 - 1)d \\ 39 &= 12 + 9d \\ d &= 3. \end{aligned}$$

Mean Property

In any arithmetic sequence, each term (except the first and last) is the average of its neighbors:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

Proof:

$$\begin{aligned} a_n &= a_{n-1} + d, \\ a_n &= a_{n+1} - d. \end{aligned}$$

Adding these, we get

$$2a_n = a_{n-1} + a_{n+1},$$

hence

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

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Example In $4,\ 7,\ 10$, the middle term $7 = \frac{4+10}{2}$.

Finding the Common Difference

If two terms a_m and a_n are known, the common difference d can be found by

$$d = \frac{a_m - a_n}{m - n}.$$

Example If $a_5 = 20$ and $a_{11} = 50$, then

$$d = \frac{50 - 20}{11 - 5} = \frac{30}{6} = 5.$$

Sum of an Arithmetic Sequence

The sum of the first n terms of an arithmetic sequence is

$$S_n = a_1 + a_2 + \cdots + a_n = n \cdot \frac{a_1 + a_n}{2}.$$

Proof: Write the sum in two ways:

$$S_n = a_1 + a_2 + \cdots + a_n,$$
$$S_n = a_n + a_{n-1} + \cdots + a_1.$$

Adding them term by term and noticing that $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$:

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots = n(a_1 + a_n),$$

so

$$S_n = n \cdot \frac{a_1 + a_n}{2}.$$

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Examples

1. Find the sum of the first 50 odd numbers.

   Solution: The sequence is $1, 3, 5, \ldots$. We can see that it's an arithmetic sequence with common difference $d = 2$.

   To find a_{50}, note that the n-th term of an arithmetic sequence is

   $$a_n = a_1 + (n-1)d.$$

   We know $a_1 = 1$ and $d = 2$, so

   $$a_{50} = a_1 + 49d = 1 + 49 \cdot 2 = 99.$$

   Now we can find the sum:

   $$S_{50} = n \cdot \frac{a_1 + a_n}{2} = 50 \cdot \frac{a_1 + a_{50}}{2} = 50 \cdot \frac{1 + 99}{2} = 50 \cdot 50 = 2500.$$

2. Find the sum $2 + 4 + 6 + \cdots + 100$.

   Solution: Here $a_1 = 2$, $a_n = 100$, and the common difference is $d = 2$. How many terms are in this sequence? Use $a_n = a_1 + (n-1)d$:

   $$100 = 2 + (n-1) \cdot 2.$$

   $$98 = 2(n-1) \quad \Rightarrow \quad n - 1 = 49 \quad \Rightarrow \quad n = 50.$$

   Therefore,

   $$S_{50} = n \cdot \frac{a_1 + a_n}{2} = 50 \cdot \frac{2 + 100}{2} = 50 \cdot 51 = 2550.$$

Common Mistakes

- Using $a_n = a_1 + nd$ instead of $a_n = a_1 + (n-1)d$.
- Forgetting that d can be negative (the sequence can decrease).
- Mixing up a_n (the n-th term) and S_n (the sum of n terms).

Arithmetic Sequences Summary

$$a_n = a_1 + (n-1)d$$

$$S_n = n \cdot \frac{a_1 + a_n}{2}$$

$$d = \frac{a_m - a_n}{m - n}$$

- Constant difference $d \Rightarrow$ linear growth/decay.
- Each term is the average of its neighbors: $a_n = \frac{a_{n-1}+a_{n+1}}{2}$.

Classwork

1. Write the first five terms of an arithmetic sequence if (a) $a_1 = 7$, $d = 2$; (b) $a_1 = 20$, $d = -3$.
2. Determine the first two terms in the sequence

$$a_1,\ a_2,\ -9,\ -2,\ 5,\ \ldots$$

3. Find a_{15} if $a_3 = 11$ and $d = 4$.
4. In an arithmetic sequence, $a_1 = 5$ and $a_{10} = 50$. Find d and a_5.
5. The 5th term of an arithmetic sequence is 23, and the 8th term is 35. Find a_1 and d.
6. Find the sum of the first 12 terms of the arithmetic sequence $4,\ 9,\ 14,\ \ldots$
7. The 15th term of an arithmetic sequence is 0, and the 25th term is -40. Find d and the sum of the first 25 terms.
8. A student decides to save money every week: \$2 in the first week, \$4 in the second, \$6 in the third, and so on, increasing by \$2 each week.
   (a) How much money will the student save in 20 weeks?
   (b) After which week will the total savings first exceed \$500?

Solutions to Classwork

1. **Write the first five terms of an arithmetic sequence.**

   (a) $a_1 = 7$, $d = 2$:

$$7,\ 9,\ 11,\ 13,\ 15.$$

   (b) $a_1 = 20$, $d = -3$:

$$20,\ 17,\ 14,\ 11,\ 8.$$

2. **Determine the first two terms in the sequence** $a_1,\ a_2,\ -9,\ -2,\ 5,\ \ldots$

   We see that $d = (-2) - (-9) = 7$. Thus $a_2 = -9 - 7 = -16$, and $a_1 = a_2 - 7 = -23$.

$$\boxed{a_1 = -23, \quad a_2 = -16.}$$

3. **Find a_{15} if $a_3 = 11$ and $d = 4$.**

$$a_{15} = a_3 + (15 - 3)d = 11 + 12 \times 4 = 59.$$

$$\boxed{a_{15} = 59.}$$

4. **In an arithmetic sequence, $a_1 = 5$ and $a_{10} = 50$. Find d and a_5.**

$$a_{10} = a_1 + 9d \Rightarrow 50 = 5 + 9d \Rightarrow d = 5.$$

$$a_5 = a_1 + 4d = 5 + 4 \times 5 = 25.$$

$$\boxed{d = 5, \quad a_5 = 25.}$$

5. **The 5th term is 23, and the 8th term is 35. Find a_1 and d.**

$$a_8 - a_5 = (a_1 + 7d) - (a_1 + 4d) = 3d = 35 - 23 = 12 \Rightarrow d = 4.$$
$$a_1 = a_5 - 4d = 23 - 16 = 7.$$
$$\boxed{a_1 = 7, \quad d = 4.}$$

6. **Find the sum of the first 12 terms of** $4,\ 9,\ 14,\ \ldots$

$$a_1 = 4, \quad d = 5, \quad a_{12} = 4 + (12 - 1) \times 5 = 59.$$
$$S_{12} = 12 \cdot \frac{a_1 + a_{12}}{2} = 12 \cdot \frac{4 + 59}{2} = 12 \times 31.5 = 378.$$
$$\boxed{S_{12} = 378.}$$

7. **The 15th term is 0, and the 25th term is -40. Find d and S_{25}.**

$$a_{25} - a_{15} = (a_1 + 24d) - (a_1 + 14d) = 10d = -40 - 0 = -40 \Rightarrow d = -4.$$
$$a_{15} = a_1 + 14d = 0 \Rightarrow a_1 = -14d = 56.$$
$$S_{25} = 25 \cdot \frac{a_1 + a_{25}}{2} = 25 \cdot \frac{56 + (-40)}{2} = 25 \times 8 = 200.$$
$$\boxed{d = -4, \quad S_{25} = 200.}$$

8. Deposits form an arithmetic sequence with $a_1 = 2$ and common difference $d = 2$.

(a) Total after 20 weeks. The 20th deposit is $a_{20} = 2 + (20 - 1) \cdot 2 = 40$. Hence

$$S_{20} = 20 \cdot \frac{a_1 + a_{20}}{2} = 10(2 + 40) = \boxed{420}.$$

(b) First week when total exceeds \$500. For this sequence

$$S_n = n \cdot \frac{2a_1 + (n-1)d}{2} = n \cdot \frac{4 + 2n - 2}{2} = n(n+1).$$

We need $n(n+1) > 500$. The least integer n is 22, and indeed $S_{22} = 22 \cdot 23 = 506 > 500$ while $S_{21} = 21 \cdot 22 = 462 < 500$.

Answer: after week $\boxed{22}$.

Homework

1. Write the first five terms of an arithmetic sequence if $a_1 = 7$ and $d = 2$.
2. Determine the first two terms in the sequence
$$a_1,\ a_2,\ -9,\ -2,\ 5,\ \ldots$$
3. Given $a_{10} = 131$ and $d = 12$, find a_1.
4. If $a_5 = 27$ and $a_{27} = 60$, find the first term a_1 and the common difference d.
5. In the arithmetic sequence $5, 17, 29, 41, \ldots$ which term has the value 497?
6. Find the sum of the first 10 terms of the series $4, 7, 10, 13, \ldots$
7. Find the sum of the first 1000 odd numbers. Can you guess/derive what the sum of the first n odd numbers is equal to?
8. Compute the sum $2 + 4 + 6 + \cdots + 2026$.
9. The 3rd term of an arithmetic sequence is 1. The 10th term is three times the 6th term. Find the first term and the common difference. *Hint: Use the formula for a_n and set up equations.*
10. There are 25 trees planted in a line at equal distances of 5 meters, with a well 10 meters from the nearest tree. A gardener waters each tree separately, always returning to the well after each trip. Find the total distance the gardener walks.
11. Is the sum of numbers $1 + 2 + 3 + \cdots + 2025$ divisible by 2025?
12. A bag of sunflower seeds was passed around the table. The first person took 1 seed, the second took 2, the third took 3, and so on — each next person took one more seed than the previous one. It is known that during the second round, a total of 100 more seeds were taken than during the first. How many people were sitting at the table?
13. Thirty desks are numbered consecutively starting from 1. Five students are seated so that the numbers of their desks form an arithmetic progression. The sum of their desk numbers is 115. Find all possible common differences.
14. Prove that the sum of n consecutive odd natural numbers (not necessarily starting with 1!), for $n > 1$, is a composite number.

*15. Find 6 consecutive odd numbers such that their sum is the cube of a natural number.

[**Hint:** assume that a is the even number between 3rd and 4th odd numbers in the sequence you are looking for. Then the sequence is $a-5, a-3, a-1, a+1, a+3, a+5$. The sum of these terms is .... Now, you need to find the proper a, so that this sum is divisible by the cube of a natural number.]

*16. Do there exist

(a) 5,

(b) 6

prime numbers forming an arithmetic sequence?

*17. The sum of the first 20 terms of an arithmetic progression is 200, and the sum of the next 20 terms is -200. Find the sum of the first 100 terms.