

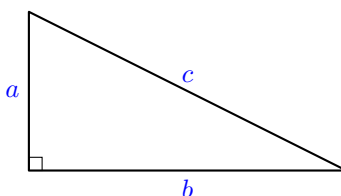
MATH 7: HANDOUT 5

RIGHT TRIANGLES AND THE PYTHAGOREAN THEOREM

The Pythagorean Theorem

In any right triangle with legs a and b and hypotenuse c , the following relation (the **Pythagorean Theorem**) holds:

$$a^2 + b^2 = c^2.$$



This theorem allows us to compute the length of one side of a right triangle if the other two are known.
Example 1: If a right triangle has legs 3 and 4, then

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25 \Rightarrow c = 5.$$

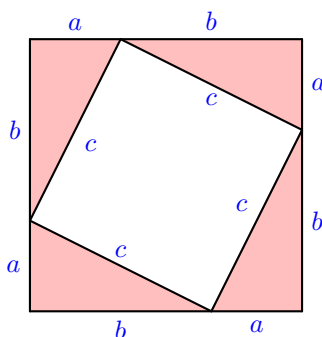
Example 2: If a right triangle has a leg 10 and the hypotenuse 11, then

$$11^2 = 10^2 + b^2 \Rightarrow b^2 = 121 - 100 = 21 \Rightarrow b = \sqrt{21}.$$

Proof of the Pythagorean Theorem

There are many proofs of the Pythagorean Theorem. Here is one of the most famous, using areas. Five more proofs are given below.

Step 1. Construct a square of side length $a + b$. Inside it, arrange four congruent right triangles with legs a and b and hypotenuse c as shown below.



Step 2. The large square has area

$$(a + b)^2.$$

It is composed of four right triangles (each of area $\frac{1}{2}ab$) and a smaller inner square of side c , area c^2 . So the total area is also

$$4 \cdot \frac{1}{2}ab + c^2 = 2ab + c^2.$$

Step 3. Equating the two expressions for the area gives:

$$(a + b)^2 = 2ab + c^2.$$

Simplify:

$$a^2 + 2ab + b^2 = 2ab + c^2 \Rightarrow a^2 + b^2 = c^2.$$

This completes the proof. ■

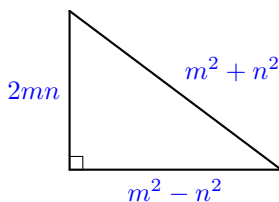
Pythagorean Triples

A **Pythagorean triple** is a set of three integers (a, b, c) that satisfy $a^2 + b^2 = c^2$. Famous examples include:

$$(3, 4, 5), \quad (5, 12, 13), \quad (8, 15, 17).$$

There is also a formula to generate Pythagorean triples (**Euclid's formula**): for integers $m > n$,

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$



It is easy to verify that such defined a, b, c satisfy the Pythagorean theorem:

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^2 \\ &= c^2 \end{aligned}$$

Examples:

1. $m = 2, n = 1$:

$$a = 2^2 - 1^2 = 4 - 1 = 3, \quad b = 2 \cdot 2 \cdot 1 = 4, \quad c = 2^2 + 1^2 = 4 + 1 = 5.$$

Triple: $(3, 4, 5)$.

2. $m = 4, n = 1$:

$$a = 4^2 - 1^2 = 16 - 1 = 15, \quad b = 2 \cdot 4 \cdot 1 = 8, \quad c = 4^2 + 1^2 = 16 + 1 = 17.$$

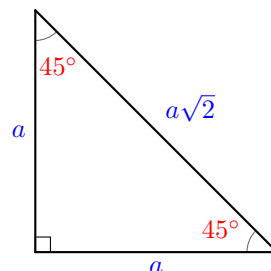
Triple: $(15, 8, 17)$.

Special Right Triangles

45-45-90 Triangle. If a right triangle has one acute angle 45° , the other is also 45° , so the legs are equal. If each leg has length a , then by Pythagorean theorem

$$c^2 = a^2 + a^2 = 2a^2 \Rightarrow c = a\sqrt{2}.$$

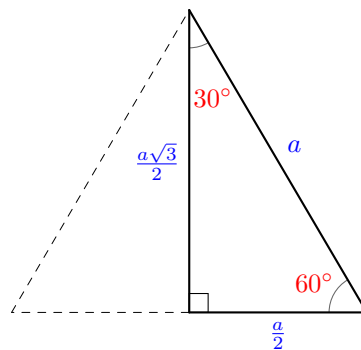
(So the side ratio is $1 : 1 : \sqrt{2}$.)



30-60-90 Triangle. A 30° – 60° – 90° triangle is half of an equilateral triangle. If the hypotenuse has length a , then the shorter leg (opposite 30°) is $\frac{a}{2}$, and the longer leg is

$$\sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{3}{4}a^2} = \frac{a\sqrt{3}}{2}.$$

(So the side ratio is $1 : \sqrt{3} : 2$.)



Five More Proofs of the Pythagorean Theorem

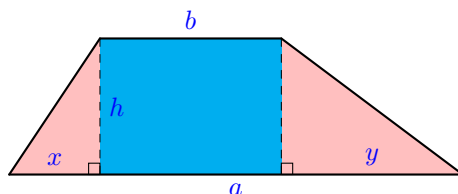
1) Garfield's Trapezoid Proof

This proof uses the formula for the area of a trapezoid.

Theorem. Let a trapezoid have parallel sides (bases) of lengths a and b and height h (the perpendicular distance between the bases). Then

$$[\text{trapezoid}] = \frac{a+b}{2} h.$$

Proof.



The trapezoid pictured above can be subdivided into a rectangle and two right triangles. Referring to the figure, we see that the area of the rectangle is bh , the area of the triangle on the left is $\frac{1}{2}xh$, and the area of the triangle on the right is $\frac{1}{2}yh$.

Thus the area of the trapezoid is

$$\begin{aligned} A &= bh + \frac{1}{2}xh + \frac{1}{2}yh \\ &= \left(b + \frac{1}{2}x + \frac{1}{2}y\right) h \\ &= \left(\frac{1}{2}(x+y) + b\right) h. \end{aligned}$$

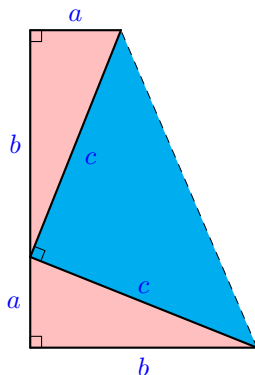
Note that $b + x + y = a$, and thus $x + y = a - b$. The area of the trapezoid is thus:

$$\begin{aligned} A &= \left(\frac{1}{2}(x+y) + b\right) h \\ &= \left(\frac{1}{2}(a-b) + b\right) h \\ &= \left(\frac{1}{2}a - \frac{1}{2}b + b\right) h \\ &= \left(\frac{1}{2}a + \frac{1}{2}b\right) h \\ &= \frac{a+b}{2} h. \end{aligned}$$

■

Now, that we have the formula for the area of a trapezoid, we can proceed to the proof of the Pythagorean Theorem.

Place two copies of the right triangle so their legs form the two bases of a right trapezoid of bases a and b . The two acute angles of the top and the bottom right triangles meet to form a right angle, so the “blue” triangle is isosceles right with legs c .



The trapezoid's height is $a + b$, therefore its area is

$$[\text{trapezoid}] = \frac{\text{base}_1 + \text{base}_2}{2} \cdot \text{height} = \frac{(a + b)}{2} \cdot (a + b) = \frac{(a + b)^2}{2}.$$

On the other hand, the trapezoid is the union of three triangles: two copies of the original (area $\frac{ab}{2}$ each) and the isosceles right triangle (area $\frac{c^2}{2}$). Thus

$$[\text{trapezoid}] = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2} = ab + \frac{c^2}{2}.$$

Equating the two area expressions and multiplying by 2:

$$(a + b)^2 = 2ab + c^2 \Rightarrow a^2 + 2ab + b^2 = 2ab + c^2 \Rightarrow a^2 + b^2 = c^2.$$

■

2) Two-Arrangement Rearrangement

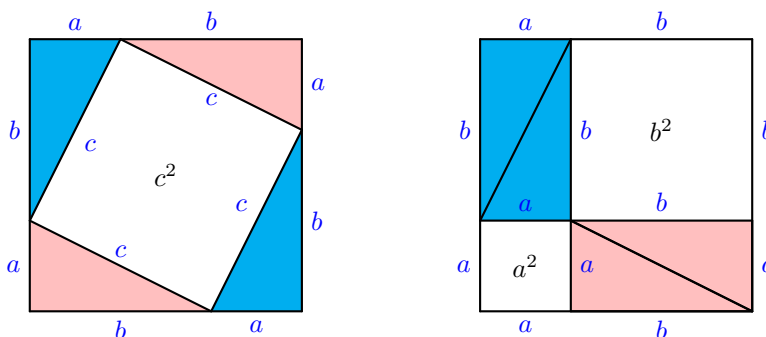
Consider the square of side $a + b$.

Arrangement 1. Place four copies of the right triangle so their hypotenuses form a central square (side c). The uncovered area is equal to c^2 .

Arrangement 2. In the *same* $a + b$ square, place “vertical” triangles in one corner with the right angle at the corner and legs along the sides, and “horizontal” triangles in the opposite corner. The uncovered region splits into two nonoverlapping squares of sides a and b , hence its area is $a^2 + b^2$.

Comparing these two arrangements, we get $c^2 = a^2 + b^2$.

■



3) No-similarity similarity proof

For this proof we need the following fact first (such auxiliary facts in mathematics are called **Lemmas**):

Lemma. If two right triangles CAB and FCE have equal acute angles, $\angle CAB = \angle FCE$, then their legs are in the same ratio:

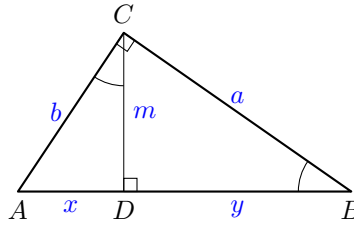
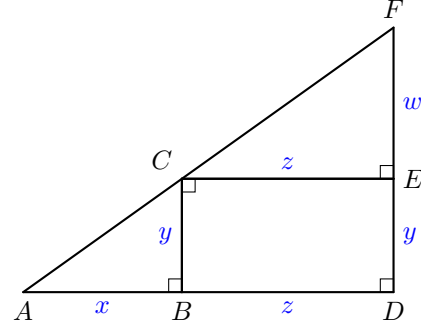
$$\frac{y}{x} = \frac{w}{z}.$$

Proof. Positioned as shown, the areas of triangles CAB and FCE together with the area of rectangle $BCED$ add up to the area of triangle AFD :

$$\frac{xy}{2} + \frac{zw}{2} + yz = \frac{(x+z)(y+w)}{2}.$$

Simplifying gives $yz = xw$, which is equivalent to

Now, we can prove the Pythagorean theorem. Drop the altitude from the right angle to the hypotenuse.



There are three triangles ABC , ACD , and CBD with congruent acute angles. From the lemma above,

$$\frac{m}{x} = \frac{y}{m} = \frac{a}{b} \Rightarrow x = m \cdot \frac{b}{a} \text{ and } y = m \cdot \frac{a}{b}$$

and so

$$AB = x + y = m \left(\frac{b}{a} + \frac{a}{b} \right). \quad (1)$$

Now since the area of triangles ADC and DCB add up to the area of ACB we have after multiplying by 2 that

$$mx + my = ab \Rightarrow m(x + y) = ab \Rightarrow m = \frac{ab}{x + y}. \quad (2)$$

Hence from (1) and (3)

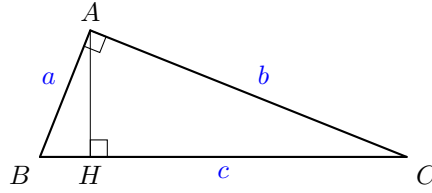
$$x + y = \frac{ab}{x + y} \left(\frac{b}{a} + \frac{a}{b} \right)$$

so that $(x + y)^2 = a^2 + b^2$ which is the Pythagorean identity. ■

The next two proofs use the concept of *similarity* of triangles. If you are unfamiliar with this concept, you can skip them.

4) Similar-Triangles (Altitude-to-the-Hypotenuse) Proof

Let H be the foot of the altitude from A to BC . Then $\triangle ABH \sim \triangle ABC$ (they share $\angle B$ and are right), and $\triangle ACH \sim \triangle ABC$ (they share $\angle C$ and are right).



Hence

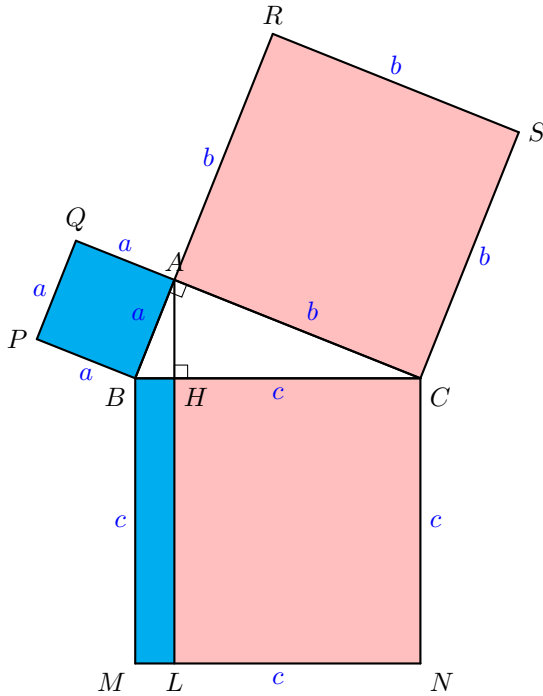
$$\frac{AB}{BC} = \frac{BH}{AB} \Rightarrow AB^2 = BH \cdot BC, \quad \frac{AC}{BC} = \frac{CH}{AC} \Rightarrow AC^2 = CH \cdot BC.$$

Adding gives

$$AB^2 + AC^2 = (BH + CH) \cdot BC = BC \cdot BC = c^2,$$

i.e. $a^2 + b^2 = c^2$. ■

5) Euclid's "Windmill" (Squares-on-the-Sides) Proof (area via similarity)



Construct squares externally on AB , AC , and BC . Draw the altitude AH to BC .

From the similarity in the first proof, we already have the *projection identities*

$$AB^2 = BH \cdot BC = BH \cdot BM = \text{Area}(BHML),$$

$$AC^2 = CH \cdot BC = CH \cdot CN = \text{Area}(CHLN).$$

Two rectangles $BHLM$ and $CHLN$ exactly partition the square on BC , adding up to its area:

$$\begin{aligned} \text{Area}(BHLM) + \text{Area}(CHLN) &= \text{Area}(BCNM) \\ &= \text{Area}(\text{square on } BC) \end{aligned}$$

Therefore the area of the square on the hypotenuse equals the sum of the areas of the squares on the legs:

$$\begin{aligned} \text{Area}(\text{square on } BC) &= \text{Area}(\text{square on } AB) \\ &\quad + \text{Area}(\text{square on } AC), \end{aligned}$$

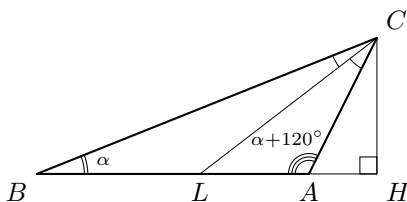
i.e. $c^2 = a^2 + b^2$. ■

Classwork

- Warm-up: Identifying Right Triangles.** Which of the following triples can form the sides of a right triangle?

(a) 6, 8, 10	(c) 9, 12, 15	(e) 4, 4, 5
(b) 5, 9, 11	(d) 7, 24, 25	(f) 8, 9, 12
- Missing Side.** In each right triangle below, find the missing side (a, b are legs, c is the hypotenuse).

(a) $a = 9, b = 12$	(c) $b = 40, c = 41$
(b) $a = 15, c = 17$	(d) $a = 5\sqrt{3}, c = 10$
- Diagonal Problems.**
 - Find the diagonal of a rectangle with sides 6 and 8.
 - Find the diagonal of a square whose side is 7.
 - A rectangular box has sides 3, 4, and 12. Find its space diagonal.
- Special Triangles.**
 - In a $45^\circ-45^\circ-90^\circ$ triangle, the hypotenuse is 10. Find the legs.
 - In a $30^\circ-60^\circ-90^\circ$ triangle, the longer leg is $6\sqrt{3}$. Find the shorter leg and hypotenuse.
- Ladder Problem.** A 13-foot ladder leans against a wall, reaching a window 12 feet above the ground. How far is the base of the ladder from the wall?
- Hexagon.** Find the area of a regular hexagon with side length 5 cm.
- Challenge.** In a triangle, one angle is 120° larger than another. Prove that the angle bisector drawn from the vertex of the third angle is twice as long as the altitude drawn from the same vertex.



Classwork — Solutions

- Warm-up: Identifying Right Triangles.** Check $a^2 + b^2 = c^2$ with c the largest.

(a) 6, 8, 10: $36 + 64 = 100 = 10^2$ — yes	(d) 7, 24, 25: $49 + 576 = 625 = 25^2$ — yes
(b) 5, 9, 11: $25 + 81 = 106 \neq 121$ — no	(e) 4, 4, 5: $16 + 16 = 32 \neq 25$ — no
(c) 9, 12, 15: $81 + 144 = 225 = 15^2$ — yes	(f) 8, 9, 12: $64 + 81 = 145 \neq 144$ — no
- Missing Side.** Use $c^2 = a^2 + b^2$.

(a) $a = 9, b = 12 \Rightarrow c = \sqrt{81 + 144} = 15$.	(c) $b = 40, c = 41 \Rightarrow a = \sqrt{1681 - 1600} = 9$.
(b) $a = 15, c = 17 \Rightarrow b = \sqrt{289 - 225} = 8$.	(d) $a = 5\sqrt{3}, c = 10 \Rightarrow b = \sqrt{100 - 75} = 5$.
- Diagonal Problems.**
 - Rectangle 6×8 : $d = \sqrt{6^2 + 8^2} = 10$.
 - Square side 7: $d = 7\sqrt{2}$.
 - Box $3 \times 4 \times 12$: $D = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$.

4. **Special Triangles.**

(a) $45^\circ - 45^\circ - 90^\circ$: legs $= \frac{10}{\sqrt{2}} = 5\sqrt{2}$.

(b) $30^\circ - 60^\circ - 90^\circ$ in ratio $1 : \sqrt{3} : 2$. If longer leg $= 6\sqrt{3} = x\sqrt{3}$, then $x = 6$. Shorter leg $= 6$, hypotenuse $= 12$.

5. **Ladder Problem.** $13^2 = 12^2 + x^2 \Rightarrow x = \sqrt{169 - 144} = 5$ ft.

6. **Hexagon.** A regular hexagon is composed of 6 equilateral triangles of side 5, each of area $\frac{1}{2} \cdot 5 \cdot \frac{5\sqrt{3}}{2} = 25 \cdot \frac{\sqrt{3}}{4}$. Hence

$$[\text{Hexagon}] = 6 \cdot 25 \cdot \frac{\sqrt{3}}{4} = \frac{75\sqrt{3}}{2} \text{ cm}^2.$$

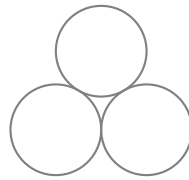
Homework

1. Rationalize the denominator.

(a) $\frac{1}{1 + \sqrt{5}}$

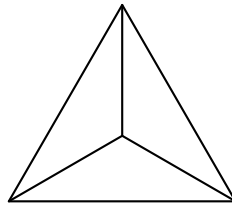
(b) $\frac{1}{1 - 2\sqrt{3}}$

2. List five different triplets of numbers that satisfy the Pythagorean theorem.
3. Find the altitude and the area of an equilateral triangle of side length 4.
4. Find the area and diagonal length of a square with side length 4.
5. A regular hexagon is inscribed in a circle of radius 6. Find the area of the hexagon.
6. Find the length of the space diagonal of a cube with side length 2.
7. Three congruent circles of radius 3 cm are stacked as shown. Find the total height of the stack.



[Hint: Consider the equilateral triangle formed by connecting the centers of the circles.]

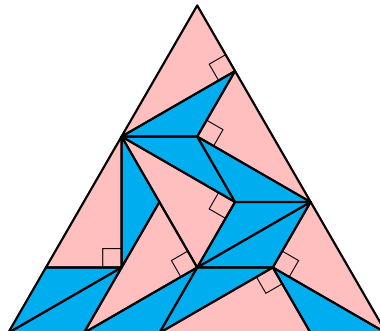
8. It is easy to cut an equilateral triangle into three congruent triangles:



Give an example of a *scalene* triangle that can be divided into three *congruent* triangles.

[Hint: We have seen this triangle in this handout!]

9. In trapezoid $ABCD$ with bases AD and BC , $\angle A = 90^\circ$ and $\angle D = 45^\circ$. It is known that $AB = 10$ cm and $AD = 3BC$. Find the area of the trapezoid.
10. In right triangle ABC , BC is the hypotenuse. Draw altitude AD perpendicular to BC , with D on BC . If $BC = 13$ and $AB = 5$, find the length of AD .
11. An equilateral triangle is made up of congruent right (pink) triangles and congruent isosceles (blue) triangles as shown in the figure below. What is the area of the equilateral triangle if the area of a blue triangle is 1? If necessary, round your answer to two decimal places.



*12. Can a 3×8 rectangle be placed inside a 5×6 rectangle?

[**Hint:** What is the longest segment that can fit inside a 5×6 rectangle?]

*13. Let AH be an altitude of triangle ABC , and let M be a point on AH . Prove that

$$AB^2 - AC^2 = MB^2 - MC^2.$$

[**Hint:** Express AB , AC , MB , and MC using Pythagorean Theorem, and compute the right and left hand sides of the identity you have to prove.]

*14. In a triangle, one angle is 120° larger than another. Prove that the angle bisector drawn from the vertex of the third angle is twice as long as the altitude drawn from the same vertex.

