MATH 7: HANDOUT 4

ALGEBRAIC EXPRESSIONS AND IDENTITIES II

Algebraic Identities and Applications

Algebraic identities are formulas that are true for all values of the variables. They allow us to expand and factor expressions quickly, often without writing out all the multiplications.

Basic Identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{(square of a sum)}$$
$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \text{(square of a difference)}$$
$$a^2 - b^2 = (a-b)(a+b) \qquad \text{(difference of squares)}$$

Examples:

$$(x+5)^2 = x^2 + 10x + 25, \qquad (2y-3)^2 = 4y^2 - 12y + 9, \qquad 49 - y^2 = (7-y)(7+y).$$

Working with Roots

Just as algebraic identities help us simplify polynomials, they also help with radical expressions. One particularly useful identity is the **difference of squares**:

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

This allows us to eliminate radicals when they appear in denominators. This process is called **rationalizing the denominator**.

Examples:

1. Simplify $\dfrac{1}{\sqrt{2}+1}$:

$$\frac{1}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2} - 1.$$

2. Simplify $\dfrac{3}{\sqrt{5}-2}$:

$$\frac{3}{\sqrt{5}-2} \cdot \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{3(\sqrt{5}+2)}{5-4} = 3\sqrt{5} + 6.$$

3. Simplify $\dfrac{1}{\sqrt{3}-\sqrt{2}}$:

$$\frac{1}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{3-2} = \sqrt{3} + \sqrt{2}.$$

Why We Rationalize the Denominator. Rationalizing the denominator means rewriting a fraction so that its denominator contains no square roots or other irrational numbers. We do this because expressions with whole-number denominators are easier to interpret, simplify, and compare. In algebra, having a rational denominator often makes later calculations—such as adding, subtracting, or substituting expressions—more straightforward. Historically, before calculators, rationalizing the denominator also made manual computation simpler. Even today, it remains the standard and more elegant way to write exact answers, helping keep mathematical expressions neat and consistent.

Cubes and Special Formulas

For third powers we have:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

There are also formulas for sums and differences of cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Note: there is no simple identity for the sum of two squares $(a^2 + b^2)$.
Examples:

$$x^3 + 27 = (x+3)(x^2 - 3x + 9), \qquad 8y^3 - 1 = (2y-1)(4y^2 + 2y + 1).$$

Equations and Factoring

Identities help us solve equations by rewriting expressions in factored form.
Example:

$$(x-2)(x+3) = 0 \quad \Rightarrow \quad x = 2 \text{ or } x = -3.$$

Linear equations such as $ax + b = 0$ can be solved directly:

$$3x + 5 = 0 \quad \Rightarrow \quad x = -\tfrac{5}{3}.$$

Classwork

1. Compute:
$$\frac{2025}{202520252025^2 - 202520252024 \cdot 202520252026}.$$
2. Find all pairs (x, y) such that
$$x^2 - 2000x = y^2 - 2000y, \qquad x \neq y.$$
3. Simplify using algebraic identities (do not expand all the multiplications):
   (a) $(x+7)^2 - (x-7)^2$
   (b) $(2a+3)^2 - (2a-3)^2$
   (c) $(5y+1)(5y-1)$
4. Rationalize the denominator:
   (a) $\dfrac{1}{\sqrt{3} - 1}$
   (b) $\dfrac{4}{2 + \sqrt{5}}$
5. Expand and simplify:
   (a) $(x+2)^3$
   (b) $(y-5)^3$
6. Simplify $\dfrac{1}{1+\sqrt{2}} + \dfrac{1}{1-\sqrt{2}}$.

Solutions to Classwork

1. Let $n = 202520252025$. Then the denominator becomes

$$n^2 - (n-1)(n+1) = n^2 - (n^2 - 1) = 1.$$

Hence,

$$\frac{2025}{202520252025^2 - 202520252024 \cdot 202520252026} = \frac{2025}{1} = 2025.$$

$$\boxed{2025.}$$

2. Bring all terms to one side:

$$x^2 - y^2 - 2000(x - y) = 0.$$

Factor:

$$(x - y)(x + y) - 2000(x - y) = (x - y)(x + y - 2000) = 0.$$

Since $x \neq y$, we must have

$$x + y - 2000 = 0 \implies x + y = 2000.$$

Thus all pairs (x, y) satisfying $x + y = 2000$ and $x \neq y$ are solutions.

$$\boxed{x + y = 2000.}$$

3. **Simplify using identities**

   (a) $(x+7)^2 - (x-7)^2 = (A^2 - B^2) = (A - B)(A + B)$ with $A = x + 7, B = x - 7$:

$$((x+7) - (x-7))((x+7) + (x-7)) = (14)(2x) = \boxed{28x}.$$

   (b) $(2a+3)^2 - (2a-3)^2 = ((2a+3) - (2a-3))((2a+3) + (2a-3)) = (6)(4a) = \boxed{24a}$.

   (c) $(5y+1)(5y-1) = (5y)^2 - 1 = \boxed{25y^2 - 1}$.

4. **Rationalize the denominator**

   (a) $\dfrac{1}{\sqrt{3}-1} \cdot \dfrac{\sqrt{3}+1}{\sqrt{3}+1} = \dfrac{\sqrt{3}+1}{3-1} = \boxed{\frac{\sqrt{3}+1}{2}}$.

   (b) $\dfrac{4}{2+\sqrt{5}} \cdot \dfrac{2-\sqrt{5}}{2-\sqrt{5}} = \dfrac{4(2-\sqrt{5})}{4-5} = -4(2-\sqrt{5}) = \boxed{4\sqrt{5} - 8}$.

5. **Expand and simplify**

   (a) $(x+2)^3 = x^3 + 3x^2 \cdot 2 + 3x \cdot 4 + 8 = \boxed{x^3 + 6x^2 + 12x + 8}$.

   (b) $(y-5)^3 = y^3 - 3y^2 \cdot 5 + 3y \cdot 25 - 125 = \boxed{y^3 - 15y^2 + 75y - 125}$.

6. $\dfrac{1}{1+\sqrt{2}} + \dfrac{1}{1-\sqrt{2}} = \dfrac{1-\sqrt{2}}{(1-\sqrt{2})(1+\sqrt{2})} + \dfrac{1+\sqrt{2}}{(1-\sqrt{2})(1+\sqrt{2})} = (-1+\sqrt{2}) + (-1-\sqrt{2}) = \boxed{-2}$.

Homework

1. Simplify each expression.
   (a) $\dfrac{42^2}{6^2}$
   (b) $\dfrac{6^3 \times 6^4}{2^3 \times 3^4}$
   (c) $(2^{-3} \times 2^7)^2$
   (d) $\dfrac{3^2 \times 6^{-3}}{10^{-3} \times 5^2}$
2. Expand the following expressions.
   (a) $(4a - b)^2$
   (b) $(a + 9)(a - 9)$
   (c) $(3a - 2b)^2$
3. Factor each expression.
   (a) $ab + ac$
   (b) $3a(a + 1) + 2(a + 1)$
   (c) $36a^2 - 49$
   (d) $a^9 - 27$
4. Use algebraic identities to evaluate:
   (a) $299^2 + 598 + 1$
   (b) 199^2
   (c) $51^2 - 102 + 1$
5. Compute:
$$\frac{(2015 \cdot 2035 + 100)(2005 \cdot 2045 + 400)}{2025^4}$$
6. Compute:
$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\cdots\left(1 - \frac{1}{225}\right)$$
7. Prove the following identity:
$$(a^2 + b^2)(u^2 + v^2) = (au + bv)^2 + (av - bu)^2$$
8. Find the expansions of $(a + b)^4$ and $(a - b)^4$ using previous results.
9. Rewrite each expression in the form $a + b\sqrt{3}$, where a and b are rational numbers.
   (a) $(1 + \sqrt{3})^2$
   (b) $(1 + \sqrt{3})^3$
   (c) $\dfrac{1}{1 - 2\sqrt{3}}$
   (d) $\dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$
   (e) $\dfrac{1 + 2\sqrt{3}}{\sqrt{3}}$