MATH 7: HANDOUT 3

ALGEBRAIC EXPRESSIONS AND IDENTITIES I

Main Algebraic Identities

In algebra, certain patterns come up so often that we call them **algebraic identities**. They allow us to expand and simplify expressions quickly without multiplying everything out.

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2}$$
$$a^2 - b^2 = (a-b)(a+b) \tag{3}$$

These are called the **square of a sum**, **square of a difference**, and the **difference of squares**.
Examples:

- $(x+5)^2 = x^2 + 10x + 25$
- $(2y-3)^2 = 4y^2 - 12y + 9$
- $49 - y^2 = (7-y)(7+y)$

Factoring Techniques

Factoring means rewriting an expression as a product of simpler factors. This is the reverse process of expanding. The main strategies are:

- **Taking out common factors.** Example: $6x^2 + 9x = 3x(2x+3)$.
- **Using identities.** Example: $x^2 - 25 = (x-5)(x+5)$ is the difference of squares.
- **Grouping.** Sometimes terms can be paired to reveal a common factor: $ax + ay + bx + by = (ax + ay) + (bx + by) = a(x+y) + b(x+y) = (a+b)(x+y)$.

In addition to there simple methods, algebraic identities can used to factor expressions that do not seem to fit the general pattern of these identities:

Example 1: Factoring $x^4 + y^4$

$$\begin{aligned} x^4 + y^4 &= x^4 + 2x^2y^2 + y^4 - 2x^2y^2 && \text{add and subtract } 2x^2y^2 \\ &= (x^2+y^2)^2 - (\sqrt{2}xy)^2 && \text{apply (1) to the first three terms} \\ &= (x^2+y^2-\sqrt{2}xy)(x^2+y^2+\sqrt{2}xy). && \text{apply (3)} \end{aligned}$$

Example 2: Factoring $x^2 - 4x + 3$

$$\begin{aligned} x^2 - 4x + 3 &= x^2 - 4x + 4 - 1 && \text{add and subtract } 1 \\ &= (x-2)^2 - 1 && \text{apply (2) to the first three terms} \\ &= (x-2-1)(x-2+1) && \text{apply (3)} \\ &= (x-3)(x-1). \end{aligned}$$

Solving Equations

Equations can often be solved by rewriting them in factored form. If a product of factors equals zero, then at least one of the factors must be zero (this is the **zero-product property**).
Example: $(x-2)(x+5)=0$ means either $x-2=0$ or $x+5=0$. Thus $x=2$ or $x=-5$.

More complicated equations can be simplified by:

- Moving all terms to one side to get a polynomial equal to zero.
- Factoring using the techniques above.
- Applying the zero-product property.

Example: $x^2+4x=0 \quad \Rightarrow \quad x(x+4)=0 \quad \Rightarrow \quad x=0$ or $x=-4$.

Homework

1. Simplify:
   (a) $\dfrac{\sqrt{63}}{\sqrt{7}}$
   (b) $\sqrt{200}$
   (c) $\sqrt{12} \cdot \sqrt{27}$

2. Express in the form $2^r 3^s a^m b^n$:
   (a) $12a^2b^3 \cdot (18a^4) \cdot (24ab)$
   (b) $8(3ab)^2(9a^3b^2)(12ab^4)$
   (c) $4a^3b^2 \cdot (6ab^5) \cdot (ab^2)^2$

3. Expand as sums of powers of x:
   (a) $(3x+4)^2$
   (b) $(1-5x)^2$
   (c) $(2-3x)^2$
   (d) $(1-x)^2(3-x)$
   (e) $(x+2)^2(2x-3)$

4. Factor:
   (a) a^2+6a+9
   (b) x^2-9
   (c) $4x^2-12xy+9y^2$
   (d) $(x+4)^2-(y-1)^2$
   (e) $a^2-b^2-10b-25$
   (f) a^4-16
   (g) x^4-y^4
   (h) $2x^3+6x^2-8x-24$
   (i) $3x^3-x^2y+6x^2y-2xy^2+3xy^2-y^3$
   (j) a^4+4 [Hint: add and subtract $4x^2$]

5. Solve:
   (a) $4(x+2)=3x+10$
   (b) $(x-1)(x+3)=0$
   (c) $\dfrac{x+4}{x+2}=3$
   (d) $(x-5)(x+6)=0$
   (e) $x^2-5x=0$
   (f) $x^3-2x=0$

6. A $4 \times 4 \times 4$ cubical box has 64 small cubes inside. How many of these touch a side or the bottom of the box?

7. Sarah's average on 7 math quizzes is 92. What score must she earn on the 8th quiz to raise her average to 93?