MATH 7: HANDOUT 2

EXPONENTS AND RADICALS

Exponents

If a is a real number and n is a positive integer, we define n-th power of a as:

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n\text{-times}}$$

The following properties hold of the exponents for any real number a and whole numbers m and n (and can easily be proven):

$$a^m \times a^n = a^{m+n} \qquad \text{Example: } 2^3 \cdot 2^4 = 2^7 = 128 \tag{1}$$

$$a^m \div a^n = a^{m-n} \qquad \text{Example: } \tfrac{5^6}{5^2} = 5^4 = 625 \tag{2}$$

$$(ab)^n = a^n b^n \qquad \text{Example: } (2 \cdot 3)^4 = 2^4 \cdot 3^4 = 16 \cdot 81 = 1296 \tag{3}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad \text{Example: } \left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16} \tag{4}$$

$$(a^m)^n = a^{mn} \qquad \text{Example: } (3^2)^4 = 3^8 = 6561 \tag{5}$$

In addition, we assume the following (for any $a \neq 0$):

$$a^1 = a$$

$$a^0 = 1 \qquad \text{Example: } 7^0 = 1$$

$$a^{-n} = \frac{1}{a^n} \qquad \text{Example: } 2^{-3} = \tfrac{1}{2^3} = \tfrac{1}{8}$$

Why do we assume that $a^0 = 1$? We can see it from the following argument:

$$a^1 = a^{1+0} = a^1 a^0$$

Also, $a^1 = a$. Therefore, we must have the following equality for any $a \neq 0$ (if we want to maintain the properties of exponents – and we do!):

$$a = a \cdot a^0 \qquad \Longrightarrow \qquad a^0 = 1.$$

Now, why do we assume that $a^{-n} = \frac{1}{a^n}$? We can see it from the following argument:

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1.$$

Dividing both parts by a^n, we get $a^{-n} = \frac{1}{a^n}$.

Radicals

A **root** of a number is another number which, when raised to a certain power, gives the original number. For example, the square root of 9: $\sqrt{9}$ is 3, because $3^2 = 9$. The cube root of 8: $\sqrt[3]{8}$ is 2, because $2^3 = 8$. In general, the n-th root of a: $\sqrt[n]{a}$ is a number b such that $b^n = a$.

Simplifying Radicals

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \qquad \text{Example: } \sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2} \tag{6}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\tfrac{a}{b}},\ b \neq 0 \qquad \text{Example: } \frac{\sqrt{18}}{\sqrt{2}} = \sqrt{9} = 3 \tag{7}$$

$$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b} \qquad \text{Example: } \sqrt[3]{250} = \sqrt[3]{125 \cdot 2} = 5\sqrt[3]{2} \tag{8}$$

Fractional Powers

What should the fractional powers be? Let us figure out what $a^{1/2}$ is:

$$a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1 = a.$$

So $a^{1/2} = \sqrt{a}$. Similarly, $a^{1/n} = \sqrt[n]{a}$.
More generally,

$$a^{m/n} = \sqrt[n]{a^m}. \tag{9}$$

Examples with Numbers

$$\begin{aligned}
27^{2/3} &= \sqrt[3]{27^2} = \sqrt[3]{729} = 9 \\
\sqrt{50} &= \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2} \\
\sqrt[3]{54} &= \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2} \\
\sqrt{240} &= \sqrt{16 \cdot 5 \cdot 3} = 4\sqrt{5}\sqrt{3}
\end{aligned}$$

Examples with Variables (assume positive)

$$\begin{aligned}
\sqrt{x^4} &= x^2, \\
\sqrt[3]{x^4} &= \sqrt[3]{x^3 \cdot x} = \sqrt[3]{x^3} \cdot \sqrt[3]{x} = x \cdot \sqrt[3]{x} \\
\sqrt[3]{x^5} &= x \cdot \sqrt[3]{x^2} \\
\sqrt[4]{x^7} &= x \cdot \sqrt[4]{x^3} \\
\sqrt[4]{80x^{14}} &= \sqrt[4]{16x^{12}5x^2} = \sqrt[4]{16} \cdot \sqrt[4]{x^{12}} \cdot \sqrt[4]{5} \cdot \sqrt[4]{x^2} = 2x^3 \cdot \sqrt[4]{5} \cdot \sqrt{x} = 2\sqrt[4]{5} \cdot x^3\sqrt{x}
\end{aligned}$$

Common Mistakes to Avoid

- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ (Counterexample: $\sqrt{9+16} = 5$, but $\sqrt{9} + \sqrt{16} = 7$)
- $(a+b)^2 \neq a^2 + b^2$ (Correct: $(a+b)^2 = a^2 + 2ab + b^2$ — but we will talk about it later!)

Homework

1. The difference between two numbers is $\frac{3}{10}$. If $\frac{2}{3}$ of the larger number is $\frac{1}{5}$ more than $\frac{1}{2}$ of the smaller, find the larger number.

2. Expand:
   (a) $3x(a + 2b - 4c)$
   (b) $-2y(a - ay + 2by)$
   (c) $(a^2 - 2a + 1)(a - 1)$
   (d) $(b^2 + 2b + 1)(b - 1)$
   (e) $(5x - 3y)(5x + 3y)$
   (f) $(4x^2 - x)(3x^2 - 2x + 7)$

3. Simplify the following expressions. When you have roots and radicals, reduce the expression to the product/ratio of radicals of simplest numbers. Refer to Example 1 above.
   (a) $\sqrt{75}$
   (b) $\sqrt{180}$
   (c) $\sqrt{98}$
   (d) $\sqrt[4]{128}$
   (e) $\sqrt[3]{864}$
   (f) $\sqrt[3]{2500}$

4. Calculate:
   (a) $18^{2/3} \div 6^{1/2}$
   (b) $81^{3/4} \cdot 64^{2/3}$
   (c) $125^{2/3} \div 25^{1/2}$
   (d) $49^{1/2} \cdot 32^{3/5} \div 2^{1/5}$

5. Simplify the following expressions with radicals. Assume all variables stand for positive numbers. Refer to Example 2 above.
   (a) $\sqrt{50n^6}$
   (b) $\sqrt{72x^4y^2}$
   (c) $\sqrt{27a^3b^7}$
   (d) $\sqrt{98a^7b^5}$
   (e) $\sqrt[3]{x^5}$
   (f) $\sqrt[3]{64q^8}$
   (g) $\sqrt[4]{y^9}$
   (h) $\sqrt{\frac{18p^5q^7}{32pq^2}}$

6. Two friends walk from school to the library. Alex takes 18 minutes, while Brian takes 24 minutes for the same distance. If the difference in their speeds is 1.5 km/h, how far is the library from the school?