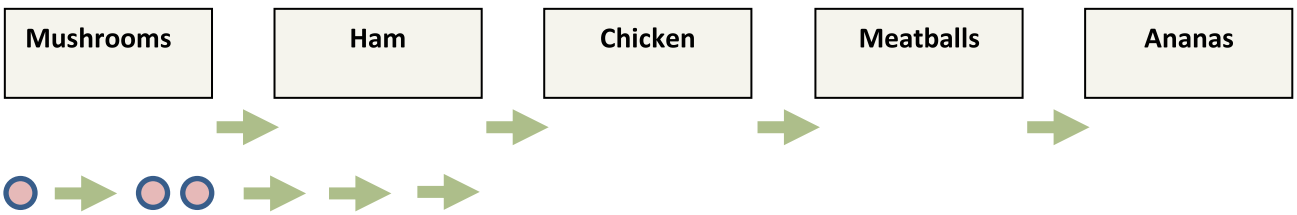
Classwork 9. Algebra.

Algebra.

The most difficult problem arise if we need to choose number of objects, order doesn't matter and repetition is allowed.
(Last week we discuss the problem, when repetition is allowed, but order was very important, for example, how many three-digit numbers can be created with digits 1, 2, 3, 4.
$4 \cdot 4 \cdot 4 = 4^3$)
As an example, we can calculate how many different ways there are to choose 3 toppings from 5 different options, with repetition allowed. For instance, you could choose three mushroom toppings, or mushroom, ham, and chicken, or two ham toppings and one chicken. Let's take a look at the problem as if we are moving from the first container to the fifth, choosing toppings along the way.



This picture illustrates the three toppings, one mushroom topping and two ham. Problem now is about rearrangement of three circles (number of toppings) and four arrows (number of moves from first container to fifth). We are choosing from $3 + 4 = 3 + (5 - 1)$

$$C_r(5,3) = \frac{\big(3 + (5-1)\big)!}{3! \cdot (3 + (5-1) - 3)!} = \frac{\big(3 + (5-1)\big)!}{3!\,(5-1)!} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 35$$

For general expression we can write:

$$C_r(n,m) = \frac{(m+n-1)!}{m!\,(n-1)!}$$

Permutations:
an arrangement of objects in a specific order. It determines the number of possible ways to order a set of distinct items, where the order of the arrangement matters. **Order matters:** The arrangement {AB} is different from {BA}.
Distinct objects: Permutations are typically calculated for sets of unique items (repetition is not allowed).

$$P(n,m) = \frac{n!}{(n-m)!}$$

Combinations:
choosing k items from a set of n distinct items. It determines the number of possible ways to choose a set of distinct items, where the order of the arrangement doesn't matter.

$$C(n,m) = \frac{n!}{(n-m)!\,m!}$$

Feature	Combinations	Permutations
Order Matters?	**No** (The order of items is irrelevant)	**Yes** (A different order creates a new result)
Formula	$C(n,m) = \frac{n!}{(n-m)!\,m!}$	$P(n,m) = \frac{n!}{(n-m)!}$
Analogy	**Selecting a group**	**Arranging a line/sequence**
Example	Choosing 3 lottery numbers ({1, 2, 3} is the same as {3, 2, 1}).	Creating a 3-digit lock code (1-2-3 is different from 3-2-1).

Repetition is allowed.
Permutations with Repetition
Since **order matters** (it's a permutation) and the selection of one item **doesn't affect** the choices for the next (repetition is allowed), the number of options for each selection remains the same.

$$P_r(n,m) = n^m$$

Where n is the size of the set and m is the number of times you choose an item (the length of the arrangement).

Examples:

In a restaurant, you can order a cheese plate. For this plate, you have to choose 4 cheeses from 10 varieties of cheese. How many different ways are there to choose your cheese plate if you must choose all different cheeses? What if you can choose the same kind of cheese several times?
Part 1. It doesn't matter how four pieces of cheese are placed on a plate, order doesn't matter, and all cheeses are different, so

$$C(10,4) = \frac{10!}{(10-4)!\,4!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1} = 10\cdot 3\cdot 7 = 210$$

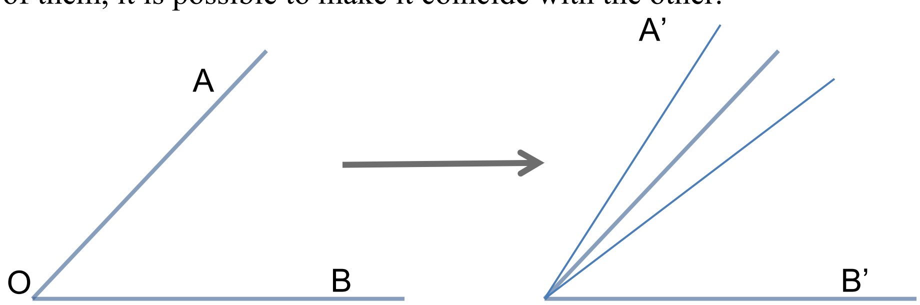
Part 2. You can choose 2 pieces of one kind of cheese and 2 pieces of another, all 4 pieces of the same kind, repetition is allowed,

$$C_r(10,4) = \frac{(4+10-1)!}{4!\,(10-1)!} = \frac{13!}{4!\cdot 9!} = 715$$

Geometry.

Congruent and non-congruent segments. Two segments are congruent if one can be placed onto the other so that their endpoints coincide. Suppose we place segment [AB] onto segment [CD] (see the picture below) by positioning point A at point C and aligning the ray [AB) with the ray [CD). If, as a result, points B and D coincide, then the segments [AB] and [CD] are congruent (equal). Otherwise, they are not congruent, and the one that fits entirely inside the other is considered smaller.

We can introduce the idea of adding several segments, as well as subtracting one segment from another.
In a similar way, we define when two angles are congruent. Two angles are congruent if, by moving one of them, it is possible to make it coincide with the other.



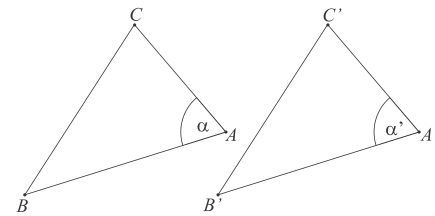
The word *equal* is often used in place of *congruent* for these objects.

- Two line segments are congruent if they have the same length.
- Two angles are congruent if they have the same measure.
- Two circles are congruent if they have the same diameter.

Sufficient conditions for triangle congruence include:

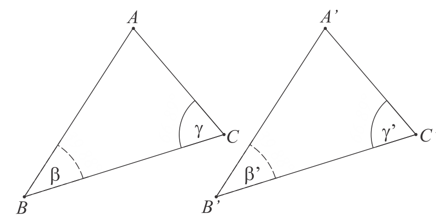
- **SAS (Side–Angle–Side):** If two pairs of sides of two triangles are equal and the included angles are equal, then the triangles are congruent.
- **SSS (Side–Side–Side):** If all three pairs of corresponding sides are equal, then the triangles are congruent.
- **ASA (Angle–Side–Angle):** If two pairs of angles and the included sides are equal, then the triangles are congruent.

SAS (Side-Angle-Side).
ABC and A'B'C' are two triangles such that



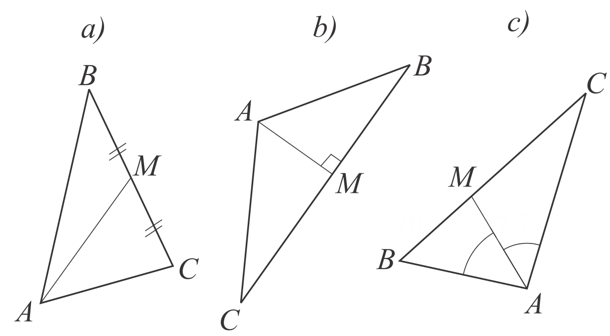
$AC = A'C', AB = A'B'$, and $\angle A = \angle A'$ We want to prove that these triangles are congruent. Superimpose ΔABC onto $\Delta A'B'C'$ in such a way that vertex A would coincide with A', the side AC would go along $A'C'$, and side AB would lie on the same side of $A'C'$ as $A'B'$. Since AC is congruent to $A'C'$, the point C will merge with point C'., due to the congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$ and due to the congruence of these sides, the point B will coincide with B'. Therefor the side BC will coincide with $B'C'$.

ASA (Angle-Side-Angle) ABC and $A'B'C'$ are two triangles, such that



$\angle C = \angle C'$, $\angle B = \angle B'$, and $BC = B'B'$.

We want to prove, that these triangles are congruent. Superimpose the triangles in such a way that point C will coincide with point C', the side CB would go along $C'B'$ and the vertex A would lie on the same side of $C'B'$ as A'. Then: since CB is congruent to $C'B'$, the point B will merge with B', and due to congruence of the angle $\angle B$ an $\angle B'$, and $\angle C$ and $\angle C'$, the side BA will go along B'A', and side CA will go along C'A'. Since two lines can intersect only at 1 point, the vertex A will have merge with A'. Thus, the triangles are identified and are congruent.



A **median** of a triangle is a segment drawn from a **vertex** to the **midpoint of the opposite side**.

An **angle bisector** is a ray (segment in a triangle) that **divides an angle into two equal angles**.

An **altitude** of a triangle is a segment drawn from a **vertex perpendicular** to the opposite side (or its extension).

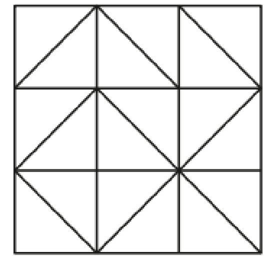
Exercises:

1. On a plane, n points are given. How many segments have their endpoints at these points?1

2. Show that

$$C(n, m) = C(n, n - m)$$

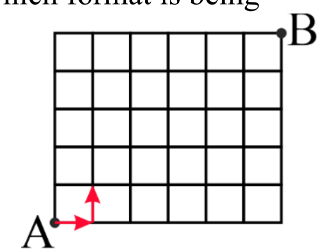
3. A square is divided into triangles (see the figure). How many ways are there to color exactly one third of the square? The small triangles cannot be colored partially.



4. Prove (show) that
   $C(n,m) - C(n,1) + C(n,2) - C(n,3) + \dots + (-1)^n C(n,n) = 0$

5. In the U.S., the date is typically written as: month number, then day number, and year. In Europe, however, the day comes first, then the month, and the year. How many days in a year have dates that cannot be interpreted unambiguously without knowing which format is being used?

6. How many different ways are there to move from point A to point B if you can only move one step to the right or one step up at a time?



7. Draw three triangles, in the first triangle, draw three medians, in the second triangle draw three bisectors, in the third triangle, draw three altitudes.

8. Prove that in the isosceles triangle bisector, drawn to the base is also altitude and median.

9. Prove that in the isosceles triangle, angles at the base are equal.

10. A line and four points A, B, C, and D, which do not lie on the line, are given. Does the line segment [AD] intersect the line if:
    a. the line segments [AB], [BC], and [CD] intersect the line;
    b. the line segments [AC] and [BC] intersect the line, but the line segment [BD] does not intersect;
    c. the line segments [AB] and [CD] intersect the line, but the line segment [BC] does not intersect;
    d. the line segments [AB] and [CD] do not intersect the line, but the line segment [BC] intersects;
    e. the line segments [AB], [BC], and [CD] do not intersect the line;
    f. the line segments [AC], [BC], and [BD] intersect the line?