Classwork 16. Algebra.

Algebra.

There are many tasks when you need to count the number of possible outcomes. For example, there are 5 chairs and 5 kids in the room. In how many ways can kids sit on these chairs? The first kid can choose any chair. The second kid can choose any of the 4 remaining chairs, the third has a choice between the three chairs, and the fifth kid has no choice at all. Therefore, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways how all of them can choose their places. Thus, obtained long expression, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, can be written as 5!. By definition:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ = 5! \qquad or \quad n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$$

This number 5! (or n!) shows the quantity of possible arrangement of 5 (n) objects, and is called permutations.

$$P = n!$$

Example 1:

1. *There are 10 books on the library shelf. How many different ways are there to place all these books on a shelf?*

Example 2:

2. *There are 10 books on the library shelf. 8 of them are authored by different authors and 2 are from the same author. How many different ways are there to place all these books on a shelf so that 2 books of one author will be next to each other?*

Because we want the two books of the same author be placed together, we can consider them as a single object and count the number of possible arrangements for 9 books, which is 9!. But in reality, for each of these arrangements, two books authored by the same author can be switch, so there are twice as many possible arrangements, $2 \cdot 9!$.

Now let's take a look on following problem:
There are 20 desks in our class and only 9 students. How many different ways are there to sit in the math class? How long it will take to try all of them, if you need 1second to switch places. First student who came in the class has 20 desks to choose the place. Second student will have only 19 choices, and so on.
The total number of possible ways to sit is

$$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 = 20 \cdot (20-1) \cdot \ldots \cdot (20-9+1) = 60949324800$$

It will take 42325920 days (or almost 116000 years) to try all of them.

In a general form we can ask about how many ways exists to arrange sample of m objects chosen out of n objects? The first object can be chosen by n different ways, second one by $(n-1)$ different way, and so on. The total number of ways will be

$$P(n,m) = n(n-1) \cdot \ldots \cdot (n-m+1)$$

This is the number of permutations of m objects chosen from n.

Can we write this formula in a shorter way?

$$P(n,m) = n(n-1)\cdot \ldots \cdot (n-m+1) = \frac{n(n-1)\cdot \ldots \cdot (n-m+1)\cdot (n-m)\cdot \ldots \cdot 3\cdot 2\cdot 1}{(n-m)\cdot \ldots \cdot 3\cdot 2\cdot 1} = \frac{n!}{(n-m)!}$$

If $m = n$, as in the first example, the formula is becoming

$$P(n,n) = n = \frac{n!}{(n-n)!} = n!$$

and it is clear that $0! = 1$, for everything to be consistent.

Example 3:

3. *In how many different ways the first three places can be awarded, if 20 people participated in the competition? In this case the repetition is not allowed, same person can't be placed in first and second place.*

Example 4:

4. *Peter has 5 final exams, LA, Math, Science, Social Studies, and Art. He can get A, B, C, and D as grades. How many different ways are there for his report card to look like? In this case we have to arrange 4 different grades in groups of 5 exams. The result of the exam can be any grade, even the same as the grade he got on another exam. For the first exam he can get any of the grade, so there is a choice of 4 grades for the first exam, as well as for any other exam.*
$$4\cdot 4\cdot 4\cdot 4\cdot 4 = 4^5$$

5. You have 5-digit lock and you forgot your code. You can check possible combinations with the speed 5 combination a minute. How long it will take you to open your locker?

Let's go back to the problem number 3 and compare it with the following problem:

6. How many different ways are there to create a team of 3 students out of 20 students of math class to take a participation in the math Olympiad. What is similar and what is different between these two problems?

The solution of the first one is

$$P(20,3) = \frac{20!}{(20-3)!} = 20\cdot 19\cdot 18 = 6840$$

Does it matter, Alice is on the first place and Robert is on the second, or vice versa? Yes, it is a big difference for them. So, the group of three winners Alice, Robert, and Lia is different from the group Robert, Alice and Lia.

If we decide to solve the problem 6 the same way:

$$\frac{20!}{(20-3)!} = 20\cdot 19\cdot 18$$

We definitely will get the group of students (Alice, Robert, and Lia) and (Robert, Alice and Lia) as different arrangements. Does it matter for the group of three students who is going to participate in the math Olympiad? Each group of three will be counted more times than needed. How many more times? Each group has $3 \cdot 2 \cdot 1 = 3! = 6$ different ways to be arranged, so we have to divide our result by this:

$$C(20,3) = \frac{20!}{(20-3)!} : 3! = \frac{20!}{(20-3)! \cdot 3!} = \frac{20 \cdot 19 \cdot 18}{6} = 1140$$

This type of choosing groups of three out of 20 and order doesn't matter, is called combinations, $C(20,3)$. Also $20C3$, or $\binom{20}{3}$ notations can be used.

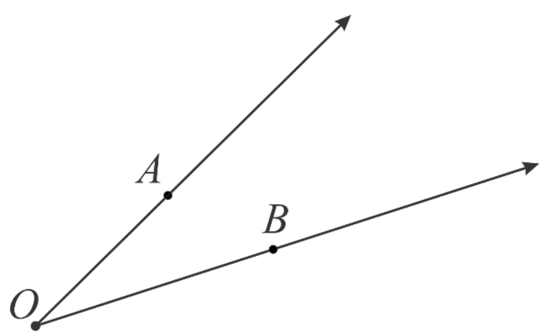
In a general way, if we want to choose set of m objects out on n objects, regardless of order

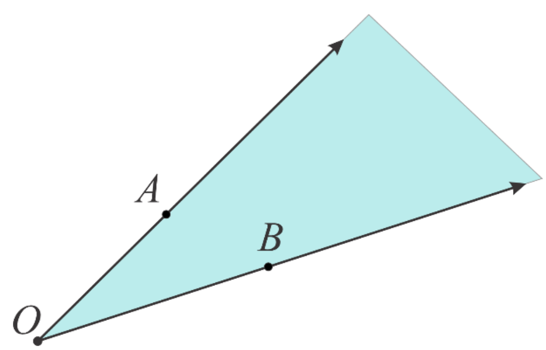
$$C(n,m) = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

Geometry.
An angle is formed by two with a common endpoint.

- *Into how many parts does an angle divide a plane?*

We can consider an angle to be two rays or two rays and the part of the plane they limit together. The difference only important when we look for the intersection of an angle and another geometrical figure.



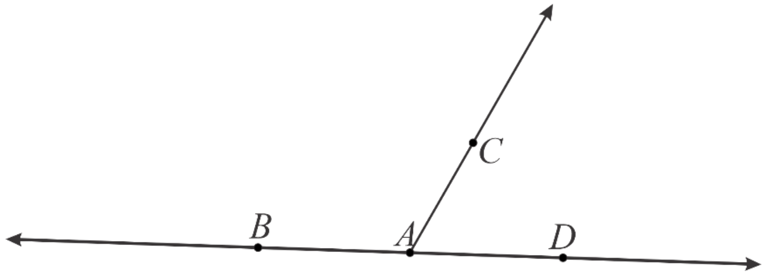


What would be the difference, if we will think about an angle as only two rays coming from the same endpoint?

Angles notations are usually three capital letters with vertex letter in the middle or small Greek letter: ∠ABC, α.

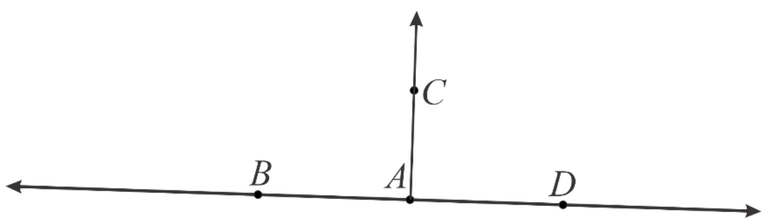
If a point marked on a line, it produces two rays with the common vertex, an angle. This angle has its own name: a straight angle.

If another ray is coming from the vertex of a straight angle, we now have three angles, $\angle CAB, \angle CAD, \angle BAD$.

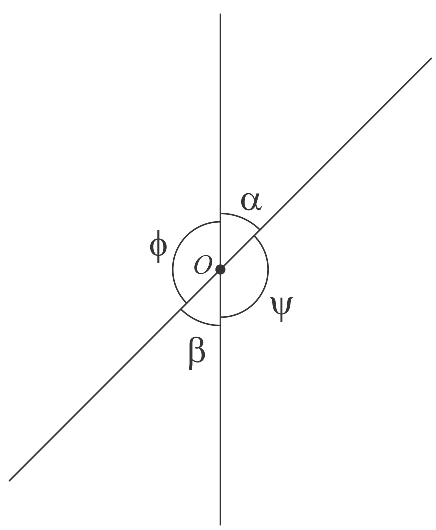


- *What can you say about these angles?*

Such angles we call supplementary angles.
There is only one angle which supplement itself to a straight angle. In this case supplementary angles are equal, and we call this angle a right angle. Measure of the straight angle is 180°, measure of the right angle is 90°



When two straight lines intersect at a point, four angles are formed. A pair of angles opposite each other formed by two intersecting straight lines that form an "X"-like shape, are called vertical angles, or opposite angles, or vertically opposite angles.



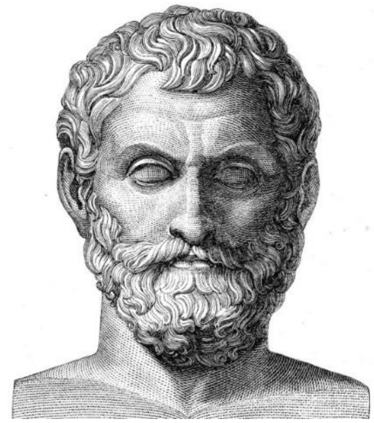
α and β and ϕ and ψ are 2 pairs of vertical angles.

Vertical angles theorem:

Vertical angles are equal.

In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements. According to a historical legend, when Thales visited Egypt, he observed that whenever the Egyptians drew two intersecting lines, they would measure the vertical angles to make sure that they were equal. Thales concluded that one could prove that vertical angles are always equal and there is no need to measure them every time.

Proof:
$\angle\phi + \angle\alpha = 180°$ because they are supplementary by construction.
$\angle\phi + \angle\beta = 180°$ because they are supplementary also by construction.
$\Rightarrow \angle\alpha = \angle\beta$, therefore we proved that if 2 angles are vertical angles then they are equal. Can we tell that invers is also the truth? Can we tell that if 2 angles are equal than they are vertical angels?



Thales of Miletus 624-546 BC was a Greek philosopher and mathematician from Miletus. Thales attempted to explain natural phenomena without reference to mythology. Thales used geometry to calculate the heights of pyramids and the distance of ships from the shore. He is the first known individual to use deductive reasoning applied to geometry; he also has been credited with the discovery of five theorems. He is the first known individual to whom a mathematical discovery has been attributed (Thales theorem).

Exercises:

1. I have 7 cookies in total, 2 oatmeal cookies, 3 chocolate chips cookies, and 2 sugar cookies. How many different ways are there to eat this all cookies?

2. Mother has 2 apples and 3 pears. Each day she gives one fruit to her kid for lunch. How many different orders are there to give these fruits?

3. 10 points are marked on the plane so that no three of them lie on the same (straight) line. How many triangles are there with vertices at these points?

4. How many different ways are there to color the table 2x2 in white and black color? 3x5?

5. There are 3 starters, 4 entrees, and 4 desserts in the price fix dinner. How many different ways are there to fix your diner?

6. Into how many parts three rays can divide a plane?

7. Into how many parts two angles can divide a plane?

8. Mark 2 points. How many different lines can be drawn through these two points?

9. Mark three points. How many lines can be drawn through three points?
   Consider all possible solution.

10. Mark four points. (Any three points do not belongs to the same line). How many lines can be drawn through four points?10 points? 100 points? N points?

11. How many intersections produce two non-parallel lines? Three non-parallel lines? 10 lines? 100? N lines?

12. 4 angles are formed at the intersection of 2 lines. One of them is 30°. What is the measure of 3 others?

13. Do the operations with angular measures:

    $a.\ 25^\circ 36' 24'' + 36^\circ 24' 40''$ $\qquad$ $b.\ 48^\circ 26' + 28^\circ 36' 34''$

    $c.\ 48^\circ 48' 48'' - 24^\circ 36' 36''$ $\qquad$ $d.\ 3 \cdot 24^\circ 36'$