Math 6c. Classwork 3.

Divisibility Rules

Divisibility tests allow us to determine whether a number is divisible by another without performing the actual division. The most common rules are:

- Every natural number is divisible by 1 and by itself.
- A number is divisible by 2 if and only if its last digit is even (0, 2, 4, 6, or 8).
- A number is divisible by 3 if and only if the sum of its digits is divisible by 3.
- A number is divisible by 4 if and only if the number formed by its last two digits is divisible by 4.
- A number is divisible by 5 if and only if its last digit is 0 or 5.
- A number is divisible by 6 if and only if it is divisible by both 2 and 3.
- A number is divisible by 7 if and only if subtracting twice the last digit from the remaining leading part of the number yields a number divisible by 7.
- A number is divisible by 8 if and only if the number formed by its last three digits is divisible by 8.
- A number is divisible by 9 if and only if the sum of its digits is divisible by 9.
- A number is divisible by 10 if and only if its last digit is 0.
- A number is divisible by 11 if and only if the alternating sum of its digits (subtracting and adding in turn) is divisible by 11.

Example:

Is 517 divisible by 11?

$5 - 1 + 7 = 11, 5 - 1 + 7 = 11, 5 - 1 + 7 = 11,$

and since 11 is divisible by 11, the original number 517 is as well.

The meaning of "if and only if"

Compare the following two statements:

- A number is divisible by 2 if its last digit is even or 0.
- A number is divisible by 2 if and only if its last digit is even or 0.

The first statement gives only a one-way implication: numbers ending in an even digit or zero are divisible by 2. It does not strictly exclude the possibility that a number ending in an odd digit might also be divisible by 2.

The second statement is precise: the condition on the last digit is both necessary and sufficient. If the last digit is even or 0, the number is divisible by 2; if the last digit is odd, it cannot be divisible by 2.

Example of proof

In earlier mathematics, properties were often "verified" only by examples. For instance, one might test whether the sum of two odd numbers is always even:

- $1 + 5 = 61 + 5 = 61 + 5 = 6$
- $25 + 29 = 5425 + 29 = 5425 + 29 = 54$
- $123 + 277 = 400123 + 277 = 400123 + 277 = 400$

In each case the sum is even, suggesting the claim is true. But examples do not constitute proof. A general proof is straightforward: any odd number can be written in the form $2k + 1$, where k is a natural number.

.If we add two odd numbers (k and m are arbitrary natural numbers):

$$2k + 1 + 2m + 1$$

We can rewrite this expression as:

$$2k + 2m + 2 = 2 \cdot (k + m + 1)$$

The result is clearly divisible by 2, hence even.

This argument shows that the statement holds universally, not just for the few numerical cases we checked. Similarly, each divisibility rule above can be rigorously proved.

Also, *to prove something* means "show". For example, to prove that $6 \cdot 5 = 30$ the definition of multiplication can be used:

$$6 \cdot 5 = 6 + 6 + 6 + 6 + 6 = 12 + 12 + 6 = 30$$

Sets and numbers.

I put a pencil, a book, a toothbrush, a coffee mug, and an apple into a bag. Do all these objects have something in common?

A set is a collection of objects that have something in common.

Can we call this collection of items a set? What is a common feature of all these objects? They are all in the bag, where I put them.

There are two ways of describing, or specifying, the members of a set. One way is by listing each member of the set, as we did with our set of things.

I can create, for example, a set of my favorite girls' names (F):

$\boldsymbol{F} = \{Mary,\ Kathrine, Sophia\}$.

The name of the set is usually indicated by a capital letter, in my case is F, list of members of the set is included in curved brackets. I can also create a set of all girls' names (N):

$\boldsymbol{N} = \{n|\ n = girl's\ name\}$

Of cause, I can't itemize all possible names, there are too many of them, we don't have enough space here for that, but I can describe the common feature of all members of the set – they are all girl's names. In the mathematical phrase above (1) I described the set, I call it N, which contains an unknown number of items, (we really don't know how many names exist), I use variable n to represent these items, girls' names. I show it by the phrase $n = girl's\ name$.

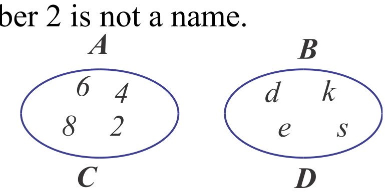
In our everyday life we use the concept of set quite often, we even have a special word for some sets, for example, the word "family" indicates a set of people, connected to each other, "class of 2020" – means all students who will graduate in 2020 and so on. Can you give more examples of such words and expressions?

When a set is created, about any object can be said that this object belongs to the set or not. For example, name "Emily" does not belong to the set F, number 2 also doesn't belong to this set. But "Emily" belongs to the set N, because it is a girl's name, number 2 is not a name.

Let's consider two examples of sets:

Sets A and B are created by listing their items explicitly:

A = {2, 4, 6, 8}, B = {d, e, s, k}.



Venn diagrams

Sets C and D are created by describing the rules according with which they were created:

C is the set of four first even natural numbers.

D is the set of letters of the word "desk".

If we look closer on our sets A and C, we can see that all elements of set A are the same as elements of set C (same goes for sets B and D).

$$\boldsymbol{A} = \boldsymbol{C} \text{ and } \boldsymbol{B} = \boldsymbol{D}$$

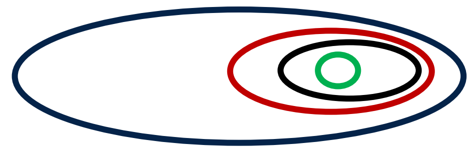
> Two sets are equal if they contain exactly the same elements.

If a set A contains element '2', then we can tell that element '2' belongs to the set A. We have a special symbol to write it down in a shorter way: $2 \in A$, $105 \notin A$. (What does this statement mean?)

When a set is created, we can say about any possible element does it belong to the given set or not.

Let's define several sets:

Set W will be the set of all words of the English language.



Set N will be the set of all nouns existing in the English language.

Set Z will be the set of all English nouns which have only 5 letters.

Set T={"table"}.

On a Venn diagram name all these sets:

If all elements of one set at the same time belong to another set then we can say that the first set is a subset of the second one. We have another special symbol to write this statement in a shorter way: ⊂.

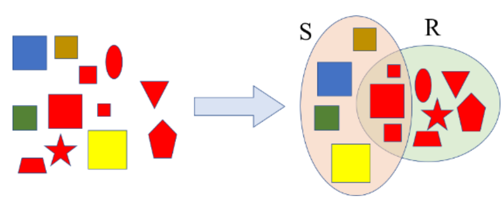
$T \subset Z \subset Y \subset W$

Problem:

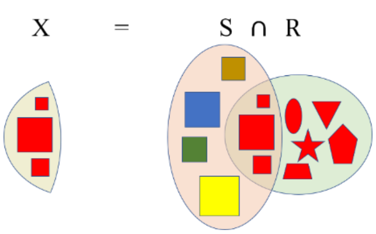
Draw Venn diagrams for the following sets:

Set A is the set of all cities of the United States. Set B is the set of all cities of the New-York state. Set C contains only one element, C = {Stony Brook}. Set D = {Paris(France), London(GB), Deli(India)}. Write the relationship between these sets.

When several sets are defined it can happen, that in accordance with all the rules we have implied, several objects can belong to several sets at the same time. For example, on a picture set S is a set of squares and a set R is a set of red shapes.



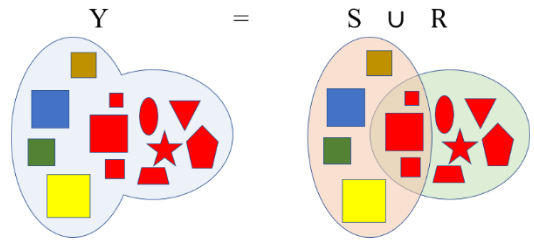
A few figures are squares and they are red, therefore they belong to both sets. Thus, we can describe a new set X containing elements that belong to the set S as well as to the set R. The new set was constructed by determining which members of two sets have the features of both sets. This statement also can be written down in a shorter version by using a special symbol ∩. Such set X is called an intersection of sets S and R.



$$\boldsymbol{X = S \cap R}$$

Another new set can be created by combining all elements of either sets (in our case S and R). Using symbol ∪ we can easily write the sentence: Set Y contains all elements of set S and set R:

$\boldsymbol{Y} = \boldsymbol{S} \cup \boldsymbol{R}$

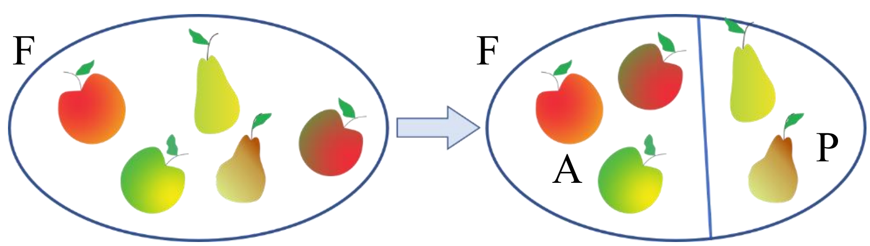


Such set Y is called a union of set S and R.

Set which does not have any element called an empty set (in math people use symbol $\varnothing$). For example, the set of polar bears living in the Antarctica is an empty set (there is no polar bears in Antarctica).

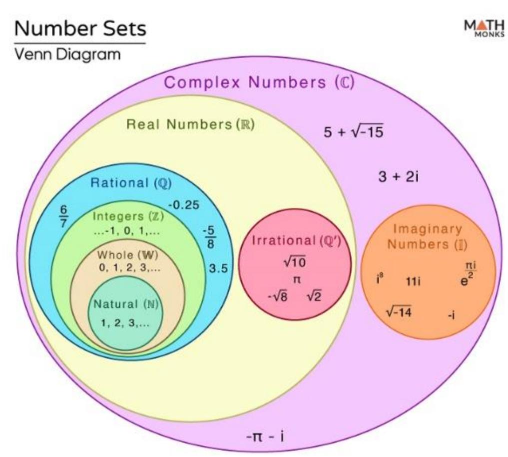
$\in$	element belongs to a set	$\not\subset$	one set is not a subset of another set
$\notin$	element does not belong to a set	$\cap$	intersection of two sets
$\subset$	one set is a subset of another set	$\cup$	union of two sets
$\varnothing$	empty set		

There is a basket with fruits on the table. It contains 3 apples and 2 pears. The set of fruits in the basket can be divided into two sets so, that these two sets do not intersect (apple can't be a pear at the same time), also in there is no any other elements in the set:



In this case we can say, we "classified" the set, we did the classification of the elements of the set.

$$\mathrm{A} \subset \mathrm{F}, \qquad \mathrm{P} \subset \mathrm{F}, \qquad \mathrm{A} \cap \mathrm{P} = \emptyset, \qquad \mathrm{A} \cup \mathrm{P} = \mathrm{F}$$



The operation of addition and subtraction of sets can noe be introduced.

$$\mathrm{F} = \mathrm{A} + \mathrm{P}, \qquad \mathrm{A} = \mathrm{F} - \mathrm{P}, \qquad \mathrm{P} = \mathrm{F} - \mathrm{A}$$

We are going to use the word "set" a lot in mathematics as well as the symbols we have just learned.
All numbers can be organized as series of sets.

$$N \subset W \subset Z \subset Q, \quad Q + Q' = R, \;\; R \subset C$$

We can do classification of sets of numbers, one of the very simple classifications of natural numbers is a classifications of by their remainders upon division by a number. All natural numbers are either even or odd, even numbers have remainder 0, when divided by 2, odd numbers have remainder 1.

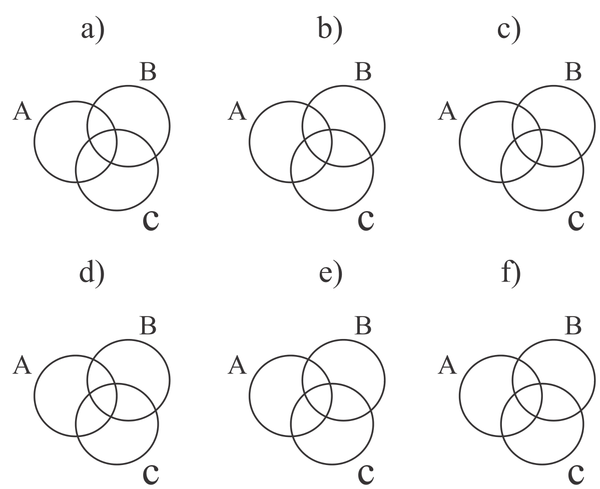
Same type of classification can be applied to natural numbers upon division by 3 and so on.

Exercises:

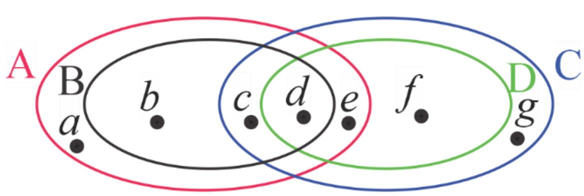
1. Prove that
   $8^5 + 2^{11}$ is divisible by 17, $9^7 - 3^{10}$ is divisible by 20
   $16^4 + 2^{12}$ is divisible by 17, $4^3 - 2^4$ is divisible by 20

2. Rewrite without parenthesis:
   $a.\ 5x(2x + 3);$ $b.\ (y + 2)^2;$ $c.\ (k + 1)(k + 2);$

3. On the diagrams of sets A, B, and C put 2 elements so that (just draw 2 points, or put any two letters).
   a. each set contains 2 elements
   b. set A contains 2 elements, set B contains also 2 elements, and set C contains 1 element.
   c. set A contains 2 elements, sets B and C contain 1 element each
   d. set A contains 2 elements, set B contains 1 element, and set C is an empty set
   e. set A contains 2 elements, set B contains 2 elements, and set C is an empty set
   f. each set contains 1 element



4. Let *A* be the set of numbers divisible by 5, *B* the set of numbers divisible by 10, *C* the set of numbers divisible by 3, and *D* the set of numbers divisible by 9. On the Euler-Venn diagram, points indicate the elements of the sets *A*, *B*, *C*, and *D* that are three-digit numbers. Assign possible values to the variables *a*, *b*, *c*, *d*, *e*, *f*, and *g*.



5. Among mathematicians, every seventh one is a philosopher, and among philosophers, every ninth one is a mathematician. Who are more numerous: philosophers or mathematicians?

6. In the garden at Emily and Jack's house, there were 2006 rose bushes. Jack watered half of all the bushes, and Emily watered half of all the bushes. It turned out that exactly three bushes — the most beautiful ones — were watered by both Emily and Jack. How many rose bushes were left unwatered?