Math 6c. Classwork 2.

Rational numbers.

Positive rational number is a number which can be represented as a ratio of two natural numbers:

$$a = \frac{p}{q}; \qquad p, q \in N$$

As we know such number is also called a fraction, p in this fraction is a nominator and q is a denominator. Any natural number can be represented as a fraction with denominator 1:

$$b = \frac{b}{1}; \;\; b \in N$$

Basic property of fraction: nominator and denominator of the fraction can be multiplied by any non-zero number n, resulting the same fraction:

$$a = \frac{p}{q} = \frac{p \cdot n}{q \cdot n}$$

In the case that numbers p and q do not have common prime factors, the fraction $\frac{p}{q}$ is irreducible fraction. If $p < q$, the fraction is called "proper fraction", if $p > q$, the fraction is called "improper fraction".

If the denominator of fraction is a power of 10, this fraction can be represented as a finite decimal, for example,

$\frac{37}{100} = \frac{37}{10^2} = 0.37,$ $\qquad \frac{3}{10} = \frac{3}{10^1} = 0.3,$ $\qquad \frac{12437}{1000} = \frac{12437}{10^3} = 12{,}437$

$$10^n = (2 \cdot 5)^n = 2^n \cdot 5^n$$

$$\frac{2}{5} = \frac{2}{5^1} = \frac{2 \cdot 2^1}{5^1 \cdot 2^1} = \frac{4}{10} = 0.4$$

$$\begin{array}{r} 0.875 \\ 8\,\overline{)\,7.000} \\ -6.4 \\ \hline 60 \\ -56 \\ \hline 40 \\ -40 \\ \hline 0 \end{array}$$

Therefore, any fraction, which denominator is represented by $2^n \cdot 5^m$ can be written in a form of finite decimal. This fact can be verified with the help of the long division, for example $\frac{7}{8}$ is a proper fraction, using the long division this fraction can be written as a decimal $\frac{7}{8} = 0.875$. Indeed,

$$\frac{7}{8} = \frac{7}{2 \cdot 2 \cdot 2} = \frac{7 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{7 \cdot 5^3}{2^3 \cdot 5^3} = \frac{7 \cdot 125}{(2 \cdot 5)^3} = \frac{875}{10^3} = \frac{875}{1000} = 0.875$$

```
   0.71428571...
7 | 5.000
  -0 0
   5 0
  -4 9
     10
    - 7
     30
   - 28
      20
    - 14
       60
     - 56
        40
      - 35
         50
       - 49
          10
         .....
```

Also, any finite decimal can be represented as a fraction with denominator 10^n.

$0.375 = \frac{375}{1000} = \frac{3}{8} = \frac{3}{2^3}$; $\quad 0.065 = \frac{65}{1000} = \frac{13 \cdot 5}{5^3 2^3} = \frac{13}{5^2 2^3}$;

$6.72 = \frac{672}{100} = \frac{168}{25} = \frac{168}{5^2}$; $\quad 0.034 = \frac{34}{1000} = \frac{17 \cdot 2}{5^3 2^3} = \frac{17}{5^3 2^2}$;

In other words, if the finite decimal is represented as an irreducible fraction, the denominator of this fraction will not have other factors besides 5^m and 2^n. Converse statement is also true: if the irreducible fraction has denominator which only contains 5^m and 2^n than the fraction can be written as a finite decimal. (Irreducible fraction can be represented as a finite decimal if and only if it has denominator containing only 5^m and 2^n as factors.)

If the denominator of the irreducible fraction has a factor different from 2 and 5, the fraction cannot be represented as a finite decimal. If we try to use the long division process, we will get an infinite periodic decimal.

At each step during this division, we will have a remainder. At some point during the process, we will see the remainder which occurred before. Process will start to repeat itself. On the example on the left, $\frac{5}{7}$, after 7, 1, 4, 2, 8, 5, remainder 7 appeared again, the fraction $\frac{5}{7}$ can be represented only as an infinite periodic decimal and should be written as $\frac{5}{7} = 0.\overline{714285}$. (Sometimes you can find the periodic infinite decimal written as $0.\overline{714285} = 0.(714285)$).

How we can represent the periodic decimal as a fraction?

Let's take a look on a few examples: $0.\overline{8}$, $2.35\overline{7}$, $0.\overline{0108}$.

$0.\overline{8}$. $x = 0.\overline{8}$ $10x = 8.\overline{8}$ $10x - x = 8.\overline{8} - 0.\overline{8} = 8$ $9x = 8$ $x = \frac{8}{9}$	$2.35\overline{7}$ $x = 2.35\overline{7}$ $100x = 235.\overline{7}$ $1000x = 2357.\overline{7}$ $1000x - 100x = 2357.\overline{7} - 235.\overline{7}$ $= 2122$ $x = \frac{2122}{900} = \frac{1061}{450}$	$0.\overline{0108}$ $x = 0.\overline{0108}$ $10000x = 108.\overline{0108}$ $10000x - x = 108$ $x = \frac{108}{9999} = \frac{12}{1111}$

Now consider $2.4\overline{0}$ and $2.3\overline{9}$

$x = 2.4\overline{0}$
$10x = 24.\overline{0}$
$100x = 240.\overline{0}$

$100x - 10x = 240 - 24$
$x = \frac{240 - 24}{90} = \frac{216}{90} = 2.4$

$x = 2.3\overline{9}$
$10x = 23.\overline{9}$
$100x = 239.\overline{9}$

$100x - 10x = 239 - 23$
$x = \frac{239 - 23}{90} = \frac{216}{90} = 2.4$