

Classwork 26.



Completing the full square.

Solve the equation:

$$x^2 = 25, \quad y^2 = 7; \quad x^2 - 16 = 0; \quad x^2 + 4x + 4 = 0$$

First two equations seem to be easy, we have to find a number, which square is 25 (these numbers are 5 and -5) and 7 (solutions are $\sqrt{7}$ and $-\sqrt{7}$). The third equation can be modified to equivalent one $x^2 = 16$, or we can factorize it:

$$x^2 - 16 = x^2 - 4^2 = (x - 4)(x + 4) = 0$$

This equation will be equal to 0 when either of the factors is 0.

$$x - 4 = 0, \quad x + 4 = 0, \quad x = \pm 4$$

The third equation also can be factorizing:

$$x^2 + 4x + 4 = x^2 + 2 \cdot 2x + 2^2 = (x + 2)^2 = (x + 2)(x + 2) = 0$$

We used the identity $(a + b)^2 = a^2 + 2ab + b^2$; but we can just use the distributive property:

$$x^2 + 4x + 4 = x^2 + 2x + 2x + 4 = x(x + 2) + 2(x + 2) = (x + 2)(x + 2)$$

This equation has only one root, $x = -2$.

This procedure is called completing the full square. Let's see how it works for more complicated equations, for example $x^2 + 6x + 8 = 0$

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + 2 \cdot 3 \cdot x + 8 = x^2 + 2 \cdot 3 \cdot x + 9 - 9 + 8 = (x + 3)^2 - 1 = (x + 3)^2 - 1^2 \\ &= (x + 3 + 1)(x + 3 - 1) = (x + 4)(x + 2) = 0 \end{aligned}$$

We used the identity $a^2 - b^2 = (a - b)(a + b)$. But we can factorize the polynomial directly:

$$x^2 + 6x + 8 = x^2 + 2x + 4x + 8 = x(x + 2) + 4(x + 2) = (x + 2)(x + 4) = 0$$

The equation is equal to 0, if any of the factors is 0, so the roots are -4 , and -2 .

Let's try to solve another equation:

$$\begin{aligned} 2x^2 + 8x + 30 = 0, &\Rightarrow x^2 + \frac{8}{2}x + \frac{30}{2} = 0; \Rightarrow x^2 + 2 \cdot \frac{4}{2}x + \frac{30}{2} = x^2 + 2 \cdot 2 \cdot x + 15 \\ &= x^2 + 2 \cdot 2 \cdot x + 4 - 4 + 15 = (x + 2)^2 + 15 = 0 \end{aligned}$$

Does this equation have any roots?

Another equation:

$$3x^2 + 2x - 30 = 0; \quad x^2 + 2 \cdot \frac{1}{3} \cdot x - 10 = 0;$$

$$\begin{aligned} x^2 + 2 \cdot \frac{1}{3} \cdot x - 10 &= x^2 + 2 \cdot \frac{1}{3} \cdot x + \frac{1}{9} - \frac{1}{9} - 10 = \left(x + \frac{1}{3}\right)^2 - \frac{1}{3} - 10 = \left(x + \frac{1}{3}\right)^2 - \frac{31}{3} \\ &= \left(x + \frac{1}{3}\right)^2 - \left(\sqrt{\frac{31}{3}}\right)^2 = \left(x + \frac{1}{3} + \sqrt{\frac{31}{3}}\right)\left(x + \frac{1}{3} - \sqrt{\frac{31}{3}}\right) = 0 \end{aligned}$$

$$x_1 = \frac{1}{3} - \sqrt{\frac{31}{3}}; \quad x_2 = -\frac{1}{3} + \sqrt{\frac{31}{3}}$$

Can we complete square in general form and find the general form of the root of quadratic equation?

Polynomial of the second order can be written generally as:

$$ax^2 + bx + c$$

a, b, c are real numbers and x is a variable.

First, let's move the common factor (first coefficient) out:

$$ax^2 + bx + c = a\left(\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Do you remember the algebraic identity

$$(k + z)^2 = (k + z)(k + z) = k^2 + 2kz + z^2$$

The last expression is

$$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Looks like $k^2 + 2kz + z^2$.

$$\begin{aligned} \left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= \left(x^2 + 2 \cdot \frac{1}{2} \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\ &= \left(x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right) \end{aligned}$$

We can further transform the expression:

$$\left(x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right) = \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}\right)^2 =$$

Last part is actually another identity:

$$p^2 - q^2 = (p + q)(p - q):$$

Therefore, we can rewrite it as

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right)^2 &= \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}\right)^2 = \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 \\ &= \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) \end{aligned}$$

Polynomial of the second order can be factorized as

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

a, b, c are numbers. Do you think any quadratic polynomial can be factorized?

Can we use this factorization process to find the way to solve a quadratic equation? Quadratic polynomial is an expression in the form

$$P(x) = ax^2 + bx + c$$

$P(x)$ means that the value of the expression depends of the value of x . What if we want to find such x that the polynomial $P(x)$ will have a certain value? For example, which x will bring the polynomial (I will omit the word “quadratic” now, but we remember, that we are talking about the quadratic polynomials) $x^2 - 6x + 13$ to 5. In other words, can we solve the equation:

$$x^2 - 6x + 13 = 5$$

First, let's rewrite this equation as

$$x^2 - 6x + 13 - 5 = 0$$

$$x^2 - 6x + 8 = 0$$

Such equations we are going to call the quadratic equation.

We have just proved that

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

For any a, b , and c , $a \neq 0$. If $a = 0$, the polynomial $P(x)$ is not quadratic.

$$a \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

Can be equal to 0 only if either of the expressions inside of the parenthesis is equal to 0.

If

$$x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0; \quad x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$$

$$x_1 = -\left(\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = -\left(\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We got two very similar expressions for our solutions:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the discriminant, and denoted as D .

$$D = b^2 - 4ac$$

For each quadratic polynomial we can find $D = b^2 - 4ac$. D is under the radical sign ($\sqrt{\quad}$) in the expressions for the roots of quadratic equation, so D can't be less than 0.

1. If $D > 0$, the equation has different roots
2. If $D = 0$, the second part of the expressions for x becomes 0, and both roots are the same, $-\frac{b}{2a}$, and polynomial is a full square,

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right) \left(x + \frac{b}{2a} \right) = a \left(x + \frac{b}{2a} \right)^2$$

3. If $D < 0$, then the square root of a negative number can't be a real number and the equation doesn't have any solution among the real numbers (but has solutions in a set of complex numbers; complex numbers are outside of the scope of our class, you will learn about them later.)

We can also rewrite the expression for factorization of the quadratic polynomial using the discriminant:

$$\begin{aligned} P(x) = ax^2 + bx + c &= a \left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right) \\ &= a \left(x + \frac{b - \sqrt{D}}{2a} \right) \left(x + \frac{b + \sqrt{D}}{2a} \right) \end{aligned}$$

Probability.

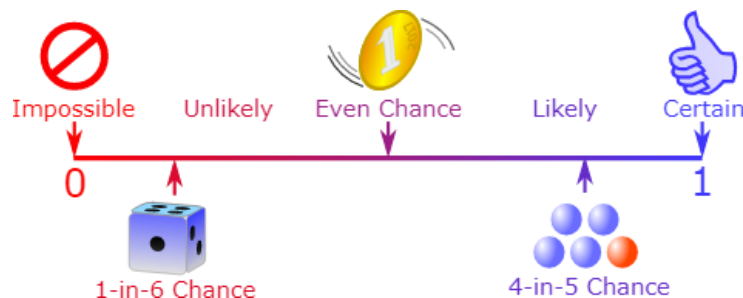
Many times a day we are saying something like

“It will probably rain today!”

“It's impossible. Tomas got an F on the test. He is a stellar student!”

“It's certain, we are flying to the Caribbean next Sunday.”

Words *possible*, *impossible*, *certain* help us to describe the likelihood of an event happening.



These words can only slightly quantify the likelihood. Can this “likelihood” be described in a more “precise” way? We can’t predict with total certainty that the event will occur. We can only say how **likely** the event is to occur using the idea of probability.

What will you get if you toss a coin? Obviously, there will be either head or tail. If we would toss this coin many times, how many heads and how many tails we will register? The ratio of the desired outcome to the total number of possible outcomes is the probability of desired outcome to happen. In the example of a tossed coin there are two possible outcomes, head and tail, so the probability to get a head is (if it’s a fair coin)



1 to 2, or $\frac{1}{2}$; or 0.5, or 50%.



It doesn’t mean that if you flip the coin twice you will definitely get a head. Rather if you toss the coin 1000 times the head will appear about 500 times. More tossing, the closer the ratio is to $\frac{1}{2}$. Let’s check it!

If we roll a die (dice can be used as singular or plural, die is used only as singular), there are 6 possible outcomes, 1, 2, 3, 4, 5, and 6.

The probability to get 1 is $\frac{1}{6}$ (there is only 1 way to get desirable outcome and 6 possible outcomes).

$$\text{Probability of an event happening} = \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$$

What is a probability to get an even number on a die?

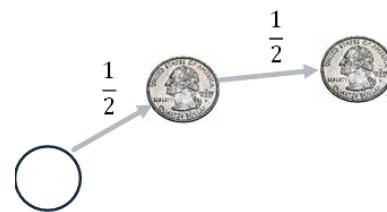
There are 3 possible ways to get even: 2, 4, 6. And 6 total outcomes, so this probability is $3:6 = 0.5$

Let’s toss a coin twice. What is a probability to get head both times?

We can look at this event (get head twice) in two different ways:

First:

Probability to get a head first is $\frac{1}{2}$.



The probability to get second head is also $\frac{1}{2}$. The probability

to get two heads in a row is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Second:

There are 4 possible outcomes for two tosses:

HH, HT, TH, TT and only one (HH) possibility for us to get HH.

What is a chance (probability) to get 6 heads in a row? If we got 5 heads, what is a probability that the next one will be a head?

Let's now play with dice.

What a chance to get double (same number on both dice)? Sum of the numbers equal to 5? 3? 6?

What is a chance to have a jackpot ticket in a lottery of 6/49? In another words, what is a probability that 6 numbers, chosen by the lottery will appear on the thicket?

The probability of the event is a ratio of numbers of the ways this event can happen to the total number of possible outcomes. In the case of lottery, there is only one winning set of six numbers. How many possible outcomes are there? We already know how to calculate it:

First number we can chose from 49, second number from 48 and so on.

To calculate the number of permutations $6P49 = 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$

General calculation of permutation

$$P(n, m) = n(n - 1)(n - 2) \cdot \dots \cdot (n - m + 1)$$

In our lottery example last factor of the expression for permutation is $44 = 49 - 6 + 1$.

$$P(n, m) = n(n - 1)(n - 2) \cdot \dots \cdot (n - m + 1) =$$

$$\frac{n(n - 1)(n - 2) \cdot \dots \cdot (n - m + 1) \cdot (n - m)(n - m - 1) \cdot \dots \cdot 3 \cdot 2 \cdot 1}{(n - m)(n - m - 1) \cdot \dots \cdot 3 \cdot 2 \cdot 1} = \frac{n!}{(n - m)!}$$

$$6P49 = \frac{49!}{(49 - 6)!}$$

Order of the numbers in the set is not important, so we have to divide the number of permutations (order matter) by the number of permutations inside the group of 6. There are exactly 6! possible way to rearrange 6 objects

$$P(6,6) = 6 \cdot 5 \cdot \dots \cdot 2 \cdot 1 = \frac{6!}{(6 - 6)!}$$

Division by 0! Mathematicians defined 0! as 1 .

$$P(6,6) = 6 \cdot 5 \cdot \dots \cdot 2 \cdot 1 = \frac{6!}{(6 - 6)!} = 6!$$

Number of possible combinations of 6 numbers from 49 is

$$C(49, 6) = \binom{49}{6} = \frac{P(49, 6)}{6!} = \frac{49!}{(49 - 6)! 6!} = \frac{49 \cdot 48 \cdot \dots \cdot 3 \cdot 2 \cdot 1}{43 \cdot 42 \cdot 41 \cdot \dots \cdot 3 \cdot 2 \cdot 1 \cdot 6 \cdot 5 \cdot \dots \cdot 2 \cdot 1}$$

$$= \frac{49 \cdot 48 \cdot \dots \cdot 44}{6 \cdot 5 \cdot \dots \cdot 2 \cdot 1} = \frac{49 \cdot 8 \cdot 47 \cdot 23 \cdot 3 \cdot 11}{1} = 13,983,816$$

Number of combinations

$$C(n, m) = \binom{n}{m} = \frac{P(n, m)}{m!} = \frac{n!}{(n - m)! m!}$$

Probability to win a jackpot is $\frac{1}{13,983,816} \approx 7.15 \cdot 10^{-8}$

Exercises:

1. Solve the quadratic equations:

a. $x^2 - 6x + 8 = 0;$

b. $x^2 - x - 2 = 0;$

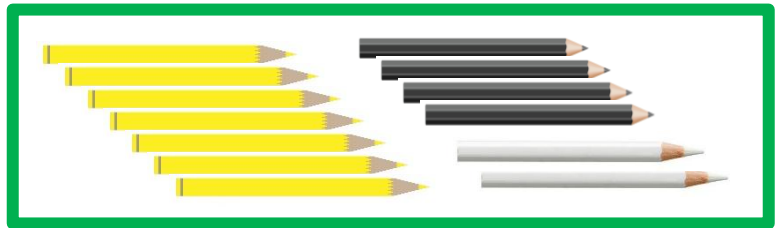
c. $5x^2 + 8x - 9 = 0$

d. $x^2 - \frac{3}{4}x + \frac{1}{8} = 0;$

e. $x^2 - \frac{1}{7}x - 50 = 0;$

f. $x^2 - 5\frac{1}{5}x + 1 = 0$

2. There are 7 yellow, two white and 4 brown pencils in a box. What is a probability to get a yellow pencil? A white pencil? A brown pencil? Two white pencils? Two yellow pencils?



3. In a bag, there are three red marbles, two blue marbles and one yellow marble, find the probability of getting

- a. red marble
- b. blue marble
- c. yellow marble

4. There are 20 students in a class, each with a different first name (there are no students with the same first name). They are all very good at math, so they decided to randomly choose a team of three to go to the math Olympiad. What is the probability that Robert, John, and Mary will be on the same team?

5. There are three boxes, each containing balls numbered from 0 to 9. One ball is taken out of each box. What is the probability that: a) three ones are drawn; b) three equal numbers are drawn?

6. A two-digit number is written at random. What is the probability that the sum of the digits of this number is equal to 5?
7. Alex wants to see how many times a "double" comes up when throwing 2 dice. After 100 trials, Alex has 19 "double" events ... is that close to what you would expect?
8. Throw a dice 3 times. What's the probability that we have three 6's?
9. From a pack of 52 cards, a card is drawn at random. What is the probability of getting a queen?
10. Throw 2 dices simultaneously. What is the probability that the summation of the numbers is multiply of 4?
11. "In a class of 25 children, two are chosen at random for duty. The probability that both of them are boys is $\frac{3}{25}$.
How many girls are in the class?"