Classwork 14.

Algebra.

We already know what is GCD and LCM for several natural numbers and we know how to find them.

Exercise:

Find GCD (GCF) and LCM for numbers

a. 222 and 345.
b. $2^2 \cdot 3^3 \cdot 5$ and $2 \cdot 3^2 \cdot 5^2$

Can we apply the same strategy to find CF and CM for algebraic expressions? (In this case the concept of GCD and LCM cannot be applied.) For example, can CF and CM be found for expressions $2x^2y^5$ and $4x^3y^2$? x and y are variables and can't be represented as a product of factors, but they itself are factors, and the expression can be represented as a product:

$2x^2y^5 = 2 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y,$ $\qquad 4x^3y^3 = 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot$

$A = (Factors, 2x^2y^5) = \{2, x, x, y, y, y, y, y\}, B = (Factors, 4x^3y^3) = \{ 2, 2, x, x, x, y, y\}$

Common devisors are any product of $A \cap B = \{ 2, x, x, y, y\}$.
What about common multiples? Product of all factors of both numbers (or the product of two numbers) will be the multiple, but minimal common multiple will be the product of the

$$A \cup B = \{2,2, x, x, x, y, y, y, y, y\}$$

$$\frac{2x^2y^5}{2 \cdot x^2y^2} = y^3; \qquad \frac{4x^3y^2}{2 \cdot x^2y^2} = 2x;$$

$$\frac{4x^3y^5}{2 \cdot x^2y^5} = 2x; \qquad \frac{4x^3y^5}{4 \cdot x^3y^3} = y^2;$$

Algebraic fractions are expressions of the form $\frac{A}{B}$ $(B \neq 0)$ (where $B \neq 0$), in which both the numerator and the denominator are algebraic expressions (not necessarily polynomials). For example:

$$\frac{3x^2 + y}{y^2 - 5x + 2}; \qquad \frac{\frac{1}{x} - 3}{y + \frac{1}{y}}$$

Properties of the algebraic fractions:

$$\frac{A}{1} = A; \qquad \frac{A}{B} = \frac{A \cdot C}{B \cdot C} \quad (C \neq 0); \qquad -\frac{A}{B} = \frac{-A}{B} = \frac{A}{-B}$$

How to algebraic fractions can be added?
Firstly, let's review the fraction addition:

$$\frac{2}{21} + \frac{5}{24}$$

The common denominator is $3 \cdot 7 \cdot 8 = 168$

$$\frac{2 \cdot 8}{168} + \frac{5 \cdot 7}{168} = \frac{16 + 35}{168} = \frac{51}{168} = \frac{17 \cdot 3}{56 \cdot 3} = \frac{17}{56}$$

Let's add

$$\frac{a}{b} \quad and \ \frac{n}{b}, \quad b \neq 0$$

These two fractions have the same denominator, b, which cannot be 0.

$$\frac{a}{b} + \frac{n}{b} = \frac{a + n}{b}$$

Another example:

$$\frac{a}{b} + \frac{c}{d}; \quad b, d \neq 0$$

Denominators are two different variables; the only possible common denominator is their product.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

Simplification of algebraic fractions:

Fraction can be reduced (simplified) if both, numerator and denominator, represented as a product, and one or more factors are common factors. It's easy, if numerator and denominator are monomials, or expressions represented as a product:

$$\frac{2a^2b}{4a^5b^2} = \frac{1}{2a^3b}, \quad a, b \ \neq 0$$

Another examples:

$$\frac{48x^3y^4z^3}{56xy^5z^4} = \frac{6x^2}{7yz}; \quad x, y, z \neq 0$$

$$\frac{2(x-1)}{5(x-1)} = \frac{2}{5}; \qquad x \neq 1;$$

$$\frac{x-y}{y-x} = -\frac{y-x}{y-x} = -1, \qquad y \neq x;$$

Irrational numbers

Rational number is a number which can be represented as a ratio of two integers:

$$a = \frac{p}{q}; \qquad p \in Z, and\ q \in N, \qquad (Z = \{\pm \cdots, \pm 1, 0\}, N = \{1, 2, \ldots\})$$

Rational numbers can be represented as infinite periodical decimals (in the case of denominators containing only powers of 2 and 5 the periodical bloc of such decimal is 0).

Numbers, which can't be express as a ratio (fraction) $\frac{p}{q}$ for any integers p and q are irrational numbers. Their decimal expansion is not finite, and not periodical.

Examples:

0.01001000100001000001…

0.123456789101112131415161718192021…

What side the square with the area of a m^2 does have? To solve this problem, we have to find the number, which gives us a as its square. In other words, we have to solve the equation

$$x^2 = a$$

This equation can be solved (has a real number solution) only if a is nonnegative (($a \geq 0$) number. It can be seen very easily;

If $x = 0, \quad x \cdot x = x^2 = a = 0,$

If $x > 0, \quad x \cdot x = x^2 = a > 0,$

If $x < 0, \quad x \cdot x = x^2 = a > 0,$

We can see that the square of any real number is a nonnegative number, or there is no such real number that has negative square.

Square root of a (real nonnegative) number a is a number, square of which is equal to a.

There are only 2 square roots from any positive number, they are equal by absolute value, but have opposite signs. The square root from 0 is 0, there is no any real square root from negative real number.

Examples:

1. Find square roots of 16: 4 and (-4), $4^2 = (-4)^2 = 16$

Arithmetic (principal) square root of a (real nonnegative) number a is a nonnegative number, square of which is equal to a.

2. Numbers $\frac{1}{7}$ and $\left(-\frac{1}{7}\right)$ are square roots of $\frac{1}{49}$, because $\frac{1}{7}\cdot\frac{1}{7}=\left(-\frac{1}{7}\right)\cdot\left(-\frac{1}{7}\right)=\frac{1}{49}$
3. Numbers $\frac{5}{3}$ and $\left(-\frac{5}{3}\right)$ are square roots of $\frac{25}{9}$, because $\left(\frac{5}{3}\right)^2=\frac{5}{3}\cdot\frac{5}{3}=\left(-\frac{5}{3}\right)^2=\left(-\frac{5}{3}\right)\cdot\left(-\frac{5}{3}\right)=\frac{25}{9}$

There is a special sign for the arithmetic square root of a number a: $\sqrt{a}$.
Examples;

1. $\sqrt{25}=5$, it means that arithmetic square root of 25 is 5, as a nonnegative number, square of which is 25. Square roots of 25 are 5 and (-5), or $\pm\sqrt{25}=\pm 5$
2. Square roots of 121 are 11 and (-11), or $\pm\sqrt{121}=\pm 11$
3. Square roots of 2 are $\pm\sqrt{2}$.
4. A few more:
   $\sqrt{0}=0$; $\sqrt{1}=1$; $\sqrt{4}=2$; $\sqrt{9}=3$; $\sqrt{16}=4$;
   $\sqrt{25}=5$; $\sqrt{\frac{1}{64}}=\frac{1}{8}$; $\sqrt{\frac{36}{25}}=\frac{6}{5}$

Base on the definition of arithmetic square root we can right

$$\left(\sqrt{a}\right)^2=a$$

To keep our system of exponent properties consistent let's try to substitute $\sqrt{a}=a^k$. Therefore,

$$\left(\sqrt{a}\right)^2=(a^k)^2=a^1$$

But we know that

$$(a^k)^2=a^{2k}=a^1 \quad \Rightarrow \quad 2k=1,\ k=\frac{1}{2}$$

To solve equation $x^2=23$ we have to find two sq. root of 23. $x=\pm\sqrt{23}$. 23 is not a perfect square as 4, 9, 16, 25, 36 …

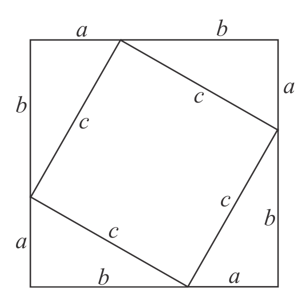
Pyphagorian theorem.

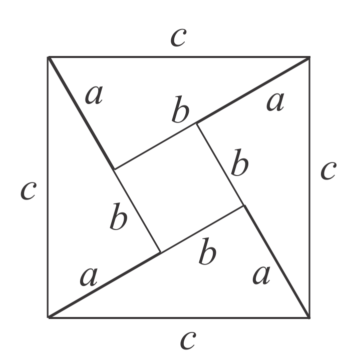
4 identical right triangles are arranged as shown on the picture. He area of the big square is $S=(a+b)\cdot(a+b)=(a+b)^2$, the are of the small square is $s=c^2$. The area of 4 triangles is $4\cdot\frac{1}{2}ab=2ab$. But also cab be represented as

$S-s=2ab$

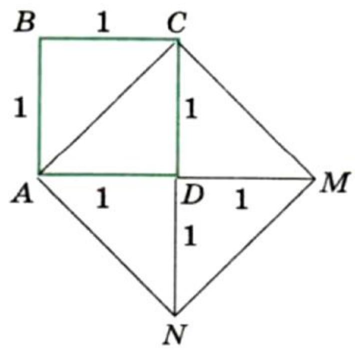
$2ab=(a+b)\cdot(a+b)-c^2=a^2+2ab+b^2-c^2$

$$\Rightarrow \qquad a^2+b^2=c^2$$





$\sqrt{2}$ is an irrational number.



The length of the segment [AC] is $\sqrt{2}$ (from Pythagorean theorem). The area of the square ACMN is twice the area of the square ABCD. Let assume that the $\sqrt{2}$ is a rational number, so it can be represented as a ratio $\frac{p}{q}$, where $\frac{p}{q}$ is nonreducible fraction.

$$\left(\frac{p}{q}\right)^2 = 2 = \frac{p^2}{q^2}$$

Or $p^2 = 2q^2$, therefore p^2 is an even number, and p itself is an even number, and can be represented as $p = 2p_1$., consequently

$p^2 = (2p_1)^2 = 4{p_1}^2 = 2q^2$

$2{p_1}^2 = q^2 \Rightarrow q$ also is an even number and can be written as $q = 2q_1$.

$\frac{p}{q} = \frac{2p_1}{2q_1}$, therefore fraction $\frac{p}{q}$ can be reduced, which is contradict the assumption. We proved that the $\sqrt{2}$ isn't a rational number by contradiction.

Exercises:

1. Add fractions:
   Example:
   $$\frac{2}{x^2a} + \frac{3}{a^2x} = \frac{2a}{a^2x^2} + \frac{3x}{a^2x^2} = \frac{2a+3x}{a^2x^2}, \quad a, x \neq 0$$

   $a.\ \frac{1}{a} + \frac{1}{b};$ $\quad b.\ \frac{2}{x} - \frac{3}{y};$ $\quad c. \frac{x}{a} + \frac{y}{b};$

   $d.\ \frac{5a}{7} - \frac{b}{x};$ $\quad e.\ \frac{1}{2a} - \frac{1}{3};$ $\quad h. \frac{1}{a} - \frac{1}{bc};$

2. Find the permissible (allowed) values of the variable in the fraction.

$$\frac{5}{x-1}; \quad \frac{3}{a-11}; \quad \frac{-7}{b+3}; \quad \frac{2a}{(a-5)(a+3)}; \quad \frac{3-x}{x^2-4}; \quad \frac{1}{(x^2-9)(x^2+25)};$$

3. For the letters M and N, select monomials such that the equality holds

   $a.\ \ 2(M-b) = 14a - 2b$ $\qquad b.\ M \cdot (2a+3b) = -6a - 9b$

   $c.\ \ N \cdot (2x - M) = 12x^2 - 18xy;$ $\qquad d.\ 3a \cdot (N + 3M) = 15abc - 3ac^2$

4. Simplify the expressions:
   a. $\left(0.3a^{n+1} - \frac{1}{12}a^n - 0.2a^{n-1}\right) \cdot 24a^n - 6a^n\left(\frac{1}{6}a^{n-1} - a^n + 0.3a^{n+1}\right)$
   b. $\left(-1\frac{1}{9}b^{n-1} + \frac{1}{3}b^n - 6b^3\right) \cdot 0.9b^{n+1} - 0.8b^n\left(\frac{7}{8}b^n - b^{n+1} - 1\frac{1}{8}b^4\right)$

5. Prove that the value of the following expressions is a rational number.

   Example:

   $(\sqrt{3}-1)(\sqrt{3}+1) = \sqrt{3}\cdot\sqrt{3} + \sqrt{3}\cdot 1 - 1\cdot\sqrt{3} - 1 = \sqrt{3}\cdot\sqrt{3} - 1 = (\sqrt{3})^2 - 1 = 3 - 1 = 2$

   $a.$ $(\sqrt{2}-1)(\sqrt{2}+1);$

   $b.$ $(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$

   $d.$ $(\sqrt{2}+1)^2 + (\sqrt{2}-1)^2$

   $d.$ $(\sqrt{2}+1)^2 + (\sqrt{2}-1)^2$

   $e.$ $(\sqrt{7}-2)^2 + 4\sqrt{7}$