MATH 6 [2026 APR 12]
HANDOUT 23 : VECTORS I

WHAT IS A VECTOR?

A vector is a geometric figure similar to a segment, but with a direction. With direction, it is possible to ADD and SUBSTRACT vectors, and even write equations for vectors, unlike for other figures.

Vectors can be denoted with two points $\overrightarrow{AB}$ and an arrow above. The first point (A) is the **beginning of the vector**, the second point (B) is the **end of the vector**. Sometimes for brewity, they can denoted with a single small letter $\vec{c}$ with an arrow above.

Two vectors are equal if, and only if

1. their **lengths** are equal
2. the lines on which they lie are **parallel**, and their **direction** (start point to end point) is the same.

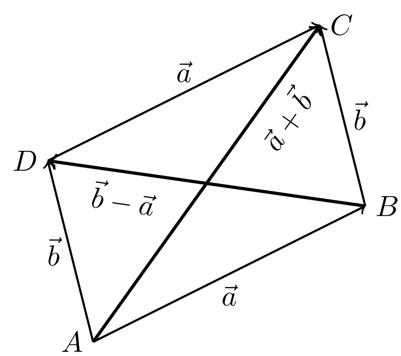
The vectors do not have to start or end at the same point to be equal.

Example In any parallelogram $ABCD$ the pairs of opposing sides are equal and parallel:

$$|AB| = |CD|\,, \qquad |AD| = |BC|\,,$$
$$(AB)\,||\,(CD)\,, \qquad (AD)\,||\,(BC)$$

Thus there are two pairs of equal vectors,

$$\overrightarrow{AB} = \overrightarrow{DC} = \vec{a}\,, \quad \overrightarrow{AD} = \overrightarrow{BC} = \vec{b}\,.$$



NOTE the direction: in the first pair, $\overrightarrow{AB}$ is equal to the vector from point D to point C. If the beginning and the end of a vector are switched, you get an **opposite vector** (having the opposite direction), and it gets the "minus" sign to show that:

$$\overrightarrow{AB} = -\overrightarrow{BA} = -\overrightarrow{CD} = \overrightarrow{DC} = \vec{a}$$

OPERATIONS WITH VECTORS

One can **construct the sum** (add vectors) by using either the triangle rule or the parallelogram rule.

Triangle rule

1. draw the first vector starting from an arbitrary point ($\overrightarrow{AB} = \vec{a}$)
2. draw the second vector starting from the end of the first vector ($\overrightarrow{BC} = \vec{b}$)
3. the vector $\overrightarrow{AB}$ (from the beginning of $\vec{a}$ to the end of $\vec{b}$) is their sum:

$$\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Parallelogram rule

1. draw the first vector starting from an arbitrary point ($\overrightarrow{AB} = \vec{a}$)
2. draw the second vector starting from the same point ($\overrightarrow{AD} = \vec{b}$).
3. complete the parallelogram by constructing lines $(DC)\,||\,(AB)$, $(BC)\|(AD)$;
   one of the diagonals is the sum, the other is the difference of the vectors:

$$\overrightarrow{AC} = \vec{a} + \vec{b}\,; \quad \overrightarrow{BD} = \vec{b} - \vec{a}\,.$$

The parallelogram rule is neat because it gives both the sum and the difference of the vectors. Also, to compute the difference ($\vec{b} - \vec{a}$), you do not need to complete the parallelogram: their difference is simply

$$\overrightarrow{BD} = \vec{b} - \vec{a} = \overrightarrow{AD}\Big(\text{vector to the end of } \overrightarrow{BD}\Big) - \overrightarrow{AB}\Big(\text{vector to the beginning of } \overrightarrow{BD}\Big)\,.$$

Adding vectors is

- commutative : $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ [clear from the parallelogram rule];
- associative : $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.

There is also the null vector (zero) denoted with $\vec{0}$: it has zero length $|\vec{0}|=0$, and for any vector $\vec{a}$,

$$\vec{a}+\vec{0}=\vec{a}\,.$$

Multiplying vector by a number

From vector $\overrightarrow{AB}$ and number x, we can construct another vector $\overrightarrow{AX}=x\,\overrightarrow{AB}$ which is

1. parallel to $\overrightarrow{AB}$;
2. has length $|AX|=|x|\,|AB|$;
3. if $x>0$, then $(x\overrightarrow{AB})\uparrow\uparrow\overrightarrow{AB}$; if $x<0$, then $(x\overrightarrow{AB})\downarrow\uparrow\overrightarrow{AB}$;
4. if $\overrightarrow{AB}=\vec{0}$ or $x=0$ (or both), then $x\overrightarrow{AB}=\vec{0}$ is the null vector.

Examples

1. If D is the midpoint of segment BC, then
$$\overrightarrow{BD}=\frac{1}{2}\overrightarrow{BC}=-\overrightarrow{CD}$$
2. in triangle ABC, the median from vertex A to side BC is
$$\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$$
Now,
$$\overrightarrow{BD}=\frac{1}{2}\overrightarrow{BC}=\frac{1}{2}\left(\overrightarrow{AC}-\overrightarrow{AB}\right)$$
therefore
$$\overrightarrow{AD}=\overrightarrow{AB}+\frac{1}{2}\left(\overrightarrow{AC}-\overrightarrow{AB}\right)=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{AC}\right)$$

Homework

1. In a triangle $\triangle ABC$, find the sum of vectors $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$.
2. Point X lies on segment AB such that $\frac{|AX|}{|BX|}=x$. Find vector $\left(\overrightarrow{AX}+\overrightarrow{BX}\right)$ if $\overrightarrow{AB}=\vec{a}$.
3. The midline of a triangle is the segment connecting the mid-points of two sides. In a triangle $\triangle ABC$, show the vector $\overrightarrow{FD}$ connecting the mid-point F of AB and the mid-point D of BC is equal to $\frac{1}{2}\overrightarrow{AC}$.
4. The midline of the trapezoid is the segment connecing the mid-points of the sides (not the bases). Show that in a trapezoid $ABCD$ with bases $AD\|BC$, the mid-line vector from the mid-point of AB to the mid-point of CD is $\frac{1}{2}\left(\overrightarrow{AD}+\overrightarrow{BC}\right)$. Can you show that it is parallel to the bases?
5. In triangle $\triangle ABC$, point P lies on side BC such that $|PC|=3|PB|$. Show that $4\overrightarrow{AP}=3\overrightarrow{AB}+\overrightarrow{AC}$.
6. Let $ABCD$ be a square and midpoints of the sides BC and CD a M and N. Let $\vec{m}=\overrightarrow{AM}$ and $\vec{n}=\overrightarrow{AN}$. Express (write formulas for) $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$ in terms of $\vec{m}$ and $\vec{n}$.