MATH 6 [2026 MAR 22]
HANDOUT 22 : GEOMETRY WITH CARTESIAN COORDINATES III:
SPECIAL POINTS IN TRIANGLES; CIRCLES AND WEDGES

PARALLEL AND PERPENDICULAR LINES

A quick reminder:

- two lines are parallel if and only if their slopes (coefficients k in equation $y = kx + b$) are equal (they can also coincide; then they also have equal intercepts);
- two lines $y = k_1x + b_1$ and $y = k_2x + b_2$ are perpendicular if and only if their slope coefficients satisfy $k_1k_2 = -1$, or $k_2 = -1/k_1$.

With that, we can find the intersection points of triangle side bisectors and triangle altitudes (*Classwork*). Using line equations, it is also easy to find the intersection point of triange medians.

Example Find the intersection of medians in $\triangle ABC$ with vertices at $A(1, 2)$, $B(6, 1)$, and $C(5, 3)$.
Solution First write equations of medians AD and BE:

- D is the midpoint of BC, so $D = \left(\frac{6+5}{2}, \frac{1+3}{2}\right) = \left(\frac{11}{2}, 2\right)$
- the line (AD) has equation $y - 2 = \frac{2-2}{11/2-1} \cdot (x - 1) = 0 \cdot (x - 1) = 0$, or $y = 2$
- similarly, the AC midpoint $E = \left(3, \frac{5}{2}\right)$, and the (BE) equation is $y - 1 = \frac{5/2-1}{3-6}(x - 6) = -\frac{1}{2}x + 3$, or $y = -\frac{1}{2} + 4$

To find the intersection $(AD) \cap (BC)$, solve these two equation together

$$\begin{cases} y = 2 \\ y = -\frac{1}{2}x + 4 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = 4 \\ y = 2 \end{cases}, \quad \text{intersection is } M(4, 2)\,.$$

To check that this point lies also on median CF, find the equation of line (CF):

$$AB \text{ midpoint } = F = \left(\frac{1+6}{2}, \frac{2+1}{2}\right) = \left(\frac{7}{2}, \frac{3}{2}\right), \quad y - 3 = \frac{3/2 - 3}{7/2 - 5}(x - 5) \Leftrightarrow y = x - 2$$

Since coordinates $(4, 2)$ of point M satisfy this equation, $M \in (CF)$.

CIRCLES

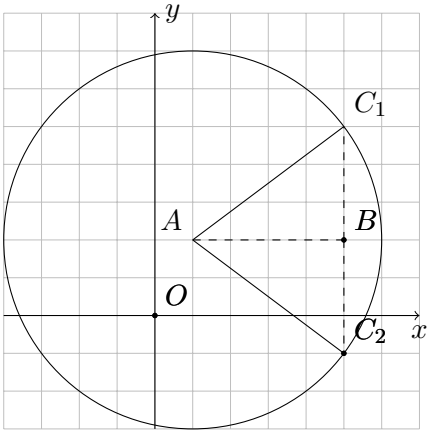
We can determine the distance between two points using the Pythagorean theorem:

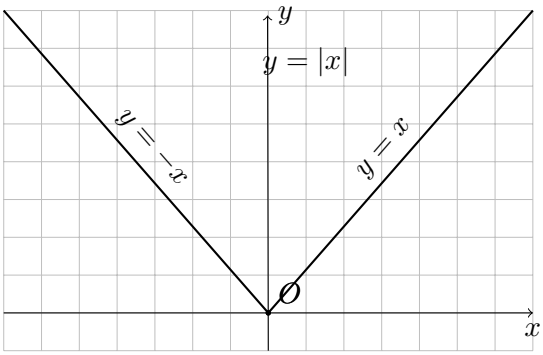
$$\text{if } A(7, 2) \text{ and } B(4, 6), \text{ then } |AB| = \sqrt{(4-7)^2 + (6-2)^2} = \sqrt{3^2 + 4^2} = 5\,.$$

If we mark one point on the plane (say $A(1, 2)$) then all the points at distance 5 will make a circle with radius $R = 5$. The equation for this circle is

$$(x - 1)^2 + (y - 2)^2 = 5^2\,, \quad \text{or } y = 2 \pm \sqrt{5^2 - (x - 1)^2}$$

(note that for some x, there are two values of y: for the lower and the upper halves of the circle).





WEDGES

In general, you can make a graph (plot) of any formula for y for each x. For example, $y = |x|$ is a "wedge", that has two semi-infinite straight pieces ("piecewise-linear") because

$$y = |x| \quad \Longleftrightarrow \left[\begin{array}{ll} y = x, & x \geq 0 \\ y = -x, & x < 0 \end{array} \right.$$

If you have graphed two equations, intersection points are the solutions of the system of these two equations (combined together).

HOMEWORK

1. In a triangle $\triangle ABC$ with vertices $A(0,0)$, $B(2,6)$, and $C(8,2)$,
   (a) find equations for the lines of altitudes (AG), (BH), and (CJ)
   ($AG \perp BC$, $BH \perp AC$, and $CJ \perp AB$);
   (b) find the coordinates of their intersection point $K = (AG) \cap (BH) \cap (CJ)$.
   Check your answer on a graph-lined paper.

2. In the same triangle,
   (a) find equations for the lines of medians (AD), (BE), and (CF)
   (D is the middle of BC, E is the middle of AC, and F is the middle of AB);
   (b) find the coordinates of their intersection point $M = (AD) \cap (BE) \cap (CF)$;
   (*c) medians are split by their intersection point 2:1 counting from the vertex; check that $|AM| = 2|MD|$, $|BM| = 2|ME|$, and $|CM| = 2|MF|$.
   Check your answer on a graph-lined paper.

3. (a) Draw the circle with equation $(x-2)^2 + y^2 - 9 = 0$; what is its center? radius?
   (b) Draw the circle with equation $x^2 + (y-1)^2 - 4 = 0$; what is its center? radius?
   (c) Draw the graph with equation $x^2 + y^2 = 0$; what kind of figure is it?

4. Sketch graphs of the following functions:

   (a) $y = |x| + 1$ (b) $y = |x + 1|$ (c) $y = |x - 5| + 1$

5. On the same patch of graph-lined paper, sketch graphs
   (a) of a circle with equation $x^2 + (y-3)^2 = 5^2$; what are its center and radius?
   (b) of a wedge with equation $y = |x| - 4$; where is its apex (corner)?
   (*c) how many solutions does the system of these two equations have,

$$\left\{ \begin{array}{l} x^2 + (y-3)^2 = 5^2 \\ y = |x| - 4 \end{array} \right. \quad ?$$