MATH 6 [2026 JAN 18]
HANDOUT 13: GEOMETRY, RULER AND COMPASS CONSTRUCTIONS

Congruence tests for triangles

Recall that by definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalities. However, it turns out that in fact, we can do with fewer checks.

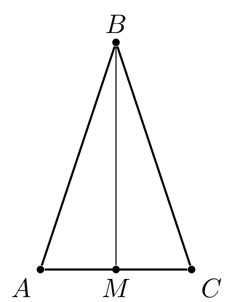
Axiom 1. *(SSS Rule). If $AB = A'B', BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 2. *(ASA Rule). If $\angle A = \angle A', \angle B = \angle B'$ and $AB = A'B'$ then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 3. *(SAS Rule). If $AB = A'B', AC = A'C'$ and $\angle A = \angle A'$ then $\triangle ABC \cong \triangle A'B'C'$.*

Isosceles Triangle

Recall that the triangle $\triangle ABC$ is called isosceles if $AB = BC$.



Theorem 1. *Properties of an isosceles triangle:*

1. *In an isosceles triangle, base angles are equal: $\angle A = \angle C$.*
2. *In an isosceles triangle, let M be the midpoint of the base AC. Then line BM is also the bisector of angle B and the altitude: BM is perpendicular to AC.* More generally, the bisector, the median, and the altitude to the base of an isosceles triangle coincide; if any two of them coincide, the triangle is isosceles.

Constructions with ruler and compass

For the next couple of classes we will be mostly interested in doing the geometric constructions with ruler and compass. Note that the ruler can only be used for drawing straight lines through two points, not for measuring distances! **Many construction problem examples can be found at** `euclidea.xyz` **(you can play without a subscription).**

When solving these problems, we need:

- Give a recipe for constructing the required figure using only ruler and compass
- **Explain why our recipe does give the correct answer**

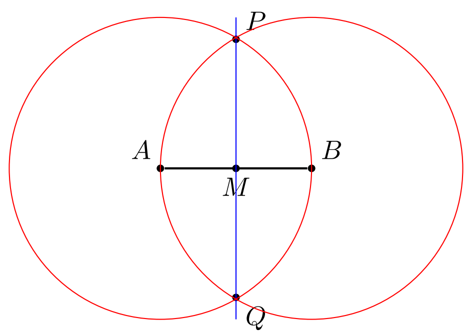
For the first part, our recipe can use only the following operations:

- Draw a line through two given points
- Draw a circle with center at a given point and given radius
- Find and label on the figure intersection points of already constructed lines and circles.

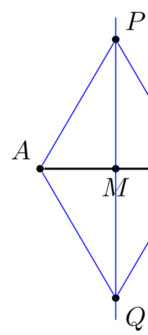
EXAMPLE: FINDING THE MIDPOINT OF THE LINE SEGMENT (PERPENDICULAR BISECTOR)

Problem: given two points A, B, construct the midpoint M of the segment AB.
Construction:

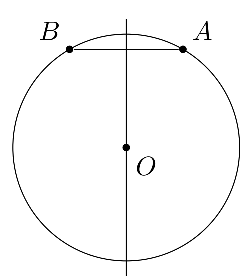
1. Draw a circle with center at A and radius AB
2. Draw a circle with center at B and radius AB
3. Mark the two intersection points of these circles by P, Q
4. Draw line through points P, Q
5. Mark the intersection point of line PQ with line AB by M. This is the midpoint.



Analysis: This is a two-step argument. In this figure, triangles $\triangle APQ$ and $\triangle BPQ$ are congruent (*why?*), so the corresponding angles are equal: From this, we can see that $\triangle APM \cong \triangle BPM$, so $AM = BM$.

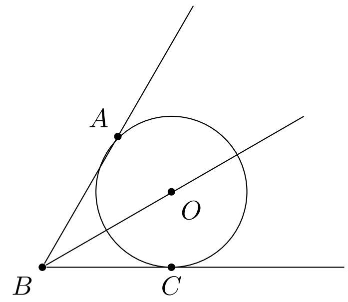


NOTE: If two points A, B are on a circle, then the center of this circle lies on perpendicular bisector to AB (i.e., a line that goes through the midpoint of AB and is perpendicular to AB).



ANGLE BISECTOR

(Recipy and proof of construction is left for the homework)
NOTE: If a circle is inscribed in the angle ABC, then the center of this circle lies on the angle bisector.



HOMEWORK

In the problems below, when lengths of figure (e.g. triangle) sides are said to be given, choose the length you like and draw a segment on your work. Make sure to use sufficient space for your construction!

Describe your recpipy briefly (e.g. "ω_1 is Circle(center at A, radius $|CD|$)", "point B at intersection of line l and circle ω_1".)

1. Given a line l and a point A on l, construct a perpendicular to l through A.
2. Given a line l and a point P outside of l, construct a perpendicular to l through P.
3. Given an angle $\angle AOB$, construct the angle bisector (i.e., a ray OM such that $\angle AOM \cong \angle BOM$).
4. Given an angle $\angle AOB$, construct angle $\angle BOC$ such that ray OB is the bisector of $\angle AOC$.
5. Given length a, construct a regular hexagon with side a.
6. Given three lengths a, b, c, construct a triangle with sides a, b, c.
7. Construct an isosceles triangle, given a base b and altitude h.
8. Construct a right triangle, given a hypotenuse h and one of the legs a.
9. Given a circle, find its center.
10. Given a triangle $\triangle ABC$, construct a circle inscribed in the triangle (see below).
11. Given a triangle $\triangle ABC$, construct a circle circumscribed around the triangle (see below).