MATH 6 [2026 JAN 11]
HANDOUT 12: PERMUTATIONS, SELECTIONS, AND REPETITIONS

CLASSWORK

Permutations and selections. Last time we discussed permutations : the number of ways n elements can be rearranged is

$$\text{Permutations(all } n \text{ elements)} = P_n = n!\,,$$

For example, there are $52!$ ways (very large number!) to shuffle a 52-card deck. Then, the number of ways k elements can be selected (in order) out of n elements is

$$\text{Permutations(}k\text{ elements out of }n\text{)} = {}_nP_k = \frac{n!}{(n-k)!}\,,$$

For example, after shuffling the deck, the top 4 cards can vary (in order) in $52!/48! = 52 \cdot 51 \cdot 50 \cdot 49$ ways.

Counting *ordered selections* of $k = 4$ cards from $n = 52$ total cards can be also imagined as

- rearranging all cards in $52!$ ways, then
- selecting the first $k = 4$ cards, and
- discarding the remaining $(n-k) = 48$ cards; their $48!$ permutations do not matter

Thus, the number of ordered selections of k from total n is

$$\underbrace{[\text{Permutations of all } n]}_{n!} = \underbrace{[\text{Permutations of } k \text{ from } n]}_{n!/(n-k)!} \cdot \underbrace{[\text{Permutations of the other } (n-k)]}_{(n-k)!}$$

$$\text{or} \quad [\text{Permutation of } k \text{ from } n] = \frac{[\text{Permutations of all } n]}{[\text{Permutations of the other } (n-k)]} = \frac{n!}{(n-k)!}$$

Permutations with repetititions. If some objects in the set are identical, they will repeat in the permutations but their order does not matter. For example, these four permutations of the nine letters in the word ALBATROSS are the same even if we tag letters “A” and “S” with a subscript:

$$\mathrm{A_1LBA_2TROS_1S_2}$$
$$\mathrm{A_2LBA_1TROS_1S_2}$$
$$\mathrm{A_1LBA_2TROS_2S_1}$$
$$\mathrm{A_2LBA_1TROS_2S_1}$$

To account for that, permutations of letters “A” and “S” only have to be divided out:

$$[\text{Permutations(ALBATROSS)}] = \frac{9!}{2!2!}$$

In the word PANAMA, there are three letters “A”; their permutations do not matter. Therefore, the number of different permutations of these six letters is

$$[\text{Permutations(PANAMA)}] = \frac{6!}{3!} = 20\,.$$

Homework (finish problems not done in class)

In combinatorics problems, you often encounter factorials, powers of large numbers, and their products. You do not need to compute them – keep the answer in the simple form with factorials and powers without actually multiplying.‘

1. How many ways can a class of 12 students into three equal teams? into four equal teams?

2. A group of 5 students decide to play a board game that needs a game master, two players with "good" alignment, and two players with "evil" alignment. In how many ways can they play?

3. How many different "words" can be made by permuting all the letters in the word ACCOUNTANT? In the word MISSISSIPPI? In the word EXCELLENCE?

4. Two chess players agree to open the game by moving three different pawns in their first three moves. In how many ways can they start the game? How many different positions can they have after that?

5. A box contains balls of different colors: two white, three black, and four red (balls are different only by color but otherwise identical).

a) In how many sequences the balls be drawn from the box?

b) What is the probability that the first drawn ball is black?

c) What is the probability that the first two balls are black?

6. About 1/6 of Americans have blue eyes. If we choose 10 people at random, what is the probability that all of them have blue eyes? that none has blue eyes? that at least one has blue eyes?

7. A puzzle consists of 9 small square pieces which must be put together to form a 3×3 square so that the pattern matches (this kind of puzzles is actually quite hard to solve!). It is known that there is only one correct solution. If you started trying all possible combinations at random, doing one new combination a second, how long will it take you to try them all?

8. At a fair, they offer you to play the following game: you are tossing small balls in a large crate full of empty bottles; if at least one of the balls lands inside a bottle, you win a stuffed toy (worth about $5). Unfortunately, it is really impossible to aim, so the game is just a matter of luck (or probability theory): every ball you toss has a 20% probability of landing inside the bottle.

a) If you are given three balls, what is the probability that all three will be hits? That all three will be misses? That at least one will be a hit?

b) Same questions for five balls.

*(c)) They charge you 2 dollars for 3 balls, or 3 dollars for 5 balls. Which is a better deal? [Considering only from the point of view of the chances of winning, not the fun you are getting]

9. In the traditional version of the domino game, the dies have two sides. Each side have between zero and six dots, and all the dies are different. How many dies are there? What if the dies have between zero and nine dots on each side?