MATH 6 [2025 NOV 23]
HANDOUT 10: COUNTING COMBINATIONS

In computing probabilities, we often need to count how many different outcomes are possible. This may be equal to how many ways there are to accomplish some task, or how many ways to select from a set, etc.

1. Combinatorics = Counting

Suppose you have to perform three tasks. If the first task can be performed in M ways, the second task can be performed in N ways, the third task can be performed in K ways, then this entire sequence of tasks can be performed in

$$\text{Count(ways to perform the three tasks)} \quad = \quad M \cdot N \cdot K \text{ ways}$$

This is elementary, but it becomes more interesting if the second and third task depends in some way on the preceding tasks. For example, if you need to schedule appointments with three different people during a week (Monday through Friday), how many different ways are there to do it? It depends on how many appointments can be scheduled in one day, and whether the order of appointments matters, and perhaps other constraints and conditions.

A If you may schedule as many appointments each day as you want (for example, all three appointments can be done on Monday) then the total number of ways to schedule them is

$$(\text{5 choices for app.1}) \cdot (\text{5 choices for app.2}) \cdot (\text{5 choices for app.3}) = 5^3 = 125\,.$$

B However, if you can schedule only one appointment each day, then there are fewer combinations because there are fewer days left for scheduling the second and the third appointments,

$$(\text{5 choices for app.1}) \cdot (\text{4 choices for app.2}) \cdot (\text{3 choices for app.3}) = 60\,.$$

C What if appointment 1 has to be the first, but appointments 2 and 3 can be in any order on different days after that? Now it gets more complicated to go through all possible choices:

$$\begin{aligned}
[\text{app.1 on Monday}] &\quad (\text{4 choices for app.2}) \cdot (\text{3 choices for app.3}) \\
[\text{app.1 on Tuesday}] &\quad + (\text{3 choices for app.2}) \cdot (\text{2 choices for app.3}) \\
[\text{app.1 on Wednesday}] &\quad + (\text{2 choices for app.2}) \cdot (\text{1 choice for app.3}) \\
&\quad = 12 + 6 + 2 = 20\,.
\end{aligned}$$

Can you figure out the number of combinations in the following scenarios? (Consider all possible cases like in example [C] above)

D In the last problem [C], what if the second appointment *has* to be on Thursday?

E What if appointments have to be on different days and in a specific order, so you have to make appointment 1 before appointment 2, and appointment 3 has to be the last?

F What if you can make *at most* two appointments per day?

G Finally, what if these appointments will be with the same person but on different days? Can be on the same day?

Such counting in scenarios like these is the section of mathematics called *Combinatorics*. In the following weeks, we will figure out efficient ways to think about and count combinations in the problems similar to the ones above. It is very important for computing probabilities: for example, if all appointment schedules are equally probable in [C] and [D] above, then naturally

$$\text{Probability that app.2 is on Thursday} = \frac{\text{Count(Schedules with app.2 on Thursday)}}{\text{Count(All possible schedules)}} = \frac{\text{Answer in [D]}}{\text{Answer in [C]}}$$

Combinatorics is also important in the branch of physics called Statistical Mechanics, which uses probability to study gases, liquids, and other media made of many (say 10^{23}) separate molecules, atoms, or other particles. There, one has to count different ways to arrange these particles. Turns out, individual particles (molecules, atoms, electrons, neutrinos) are indistinguishable, and there is no experiment possible to tell one from another. This has deep consequences for counting their arrangements and the properties of liquids and gases.

2. Homework

1. A dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses and 3 desserts. How many possible dinners are there?

2. In a certain club of 30 people, they are selecting a president, vice-president, and a treasurer (they all must be different people: no one is allowed to take two posts at once). How many ways are there to do this?

3. How many ways are there to split a class of 6 boys and 4 girls into two soccer teams fairly? (To keeps the teams fair, each team must have 3 boys and 2 girls).

4. How many whole numbers between 1–1000 are divisible by 3? by 5? by 15? are not divisible by either 3 or 5?
Hint: think of each group of numbers as sets, or set intersections, or set complements.

5. How many whole numbers between 1–100 have the sum of digits equal to 5 ? 6 ? 7?
How many whole numbers between 1–1000 have the sum of digits equal to 3 ? 4 ? 5?

6. In a group of 100 people everybody speaks either English or Chinese. Of them 73 speak English, and 43 can speak Chinese.
How many can speak English only?
How many can speak Chinese only?
How many can speak both English and Chinese?

7. The guidelines at a certain college specify that for the introductory English class, the professor may choose one of 3 specified novels, and choose two from a list of 5 specified plays. Thus, the reading list for this introductory class must have one novel and two plays. How many different reading lists could a professor create within these parameters?

8. *You may want to use a calculator for this one*

(a) In a class of 25 students, everyone chooses a date (e.g., March 13). How many combinations are possible? (Students only choose month and day, not year; February 29th is not allowed, so there are 365 different possibilities. Also, it matters who had chosen which day: combination where Bill has chosen March 12 and John, June 15 is considered different from the one where Bill has chosen June 15 and John March 12.)

(b) In the same situation, how many such combinations are possible if we additionally require that all dates must be different?

*(c) Suppose now that each of these 25 students has chosen a date at random, not knowing the choices of others. What is the probability that all of these dates will be different? That at least 2 will coincide?