MATH 6 [2025 NOV 16]
HANDOUT 9: PROBABILITY. GAMES OF CHANCE

BASIC PROBABILITY

Basic probability rule:

$$P(win) = \frac{\text{number of winning outcomes}}{\text{total number of possible outcomes}}$$

This works only if all outcomes are equally likely! Usually, a coin landing with the tail or the head is equally probable. An honest dice is equally probable to land with any of the 6 sides up. (However, a cheater's "loaded" dice may have one of the sides (which?) heavy, so that it will land on one of the sides more often.)

For example, probability of drawing a spade card out of the standard deck is

$$P = \frac{\text{number of spade cards}}{\text{total number of cards}} = \frac{13}{52} = \frac{1}{4}\,.$$

Complement rule. If the probaility of some event is P then the probability that this event **does not** happen is

$$\text{Probability(event \textbf{DOES NOT} happen)} \;=\; 1 \;-\; \text{Probability(event happens)} \;= 1 - P\,.$$

For example, if we draw a card from the deck then the probability that it is **not** a spade is $1 - \frac{1}{4} = \frac{3}{4}$. If A is a set, then the probabilities for a draw (a card, or a lottery ticket) to be or not to be from this set add up to one:

$$P(A) + P(\overline{A}) \;=\; 1\,.$$

Product rule. If we do two trials (e.g., rolling a die twice), then the probability of getting result A in the first trial and result B is the second one is

$$P(\text{A, then B}) = P(A)P(B)$$

if results of the second trial **do not depend** on the first one.

Addition rule. If we are interested in either of two outcomes that cannot happen at the same time (mutually exclusive), the probabilities add. For example, the probability to draw a king OR an ace is

$$\text{Probability(king OR ace)} \;=\; \text{Probability(king)} \;+\; \text{Probability(ace)} \;=\frac{4}{52}+\frac{4}{52}=\frac{2}{13}$$

However, if two things can happen at the same time, it is different:

$$\text{Probability(hearts OR queen)} \;=\; \text{Probability(hearts)} \;+\; \text{Probability(queen NOT hearts)} \;=\frac{1}{4}+\frac{4-1}{52}=\frac{4}{13}$$

Generally, you have to split the winning outcomes (cards) into non-intersecting subsets and then add their probabilities.

EXAMPLES

1. If toss a coin 10 times, what is the probability that all will be heads?
   Answer. $\left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$ (using calculator, one can compute that it is $1/1024 \approx 0.001$, or $1/10$ of 1%).
2. If toss a coin 10 times, what is the probability that all will be tails?
   Answer. The same.
3. If we toss a coin 10 times, what is the probability that **at least one** will be heads?
   Answer. Unfortunately, there are very many combinations which give at least one heads. In fact, it is easier to say which combinations **do not** give at least one heads: there is exactly one such combination, all tails; probability of getting this combination is, as we computed, $1/2^{10} = \frac{1}{1024}$. The remaining combinations will give at least one heads; thus probability of getting at least one heads is $1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.999$.
4. If we toss a coin 3 times, what is the probability that we get 1 tail and 2 heads?
   Answer. Since the 1 tail and 2 heads can come in any order, we have to add up these probabilities,

   $$P(\text{1 tail, 2 heads}) = P(THH) + P(HTH) + P(HHT) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{8}$$

5. What is the probability that if we roll 3 dice, all the numbers will be odd?
Alice and Bill are playing the following game. They roll 3 dice; if all numbers are odd, Alice wins, and Bill pays her \$5. Otherwise Alice loses and she pays \$1 to Bill. Would you prefer to play for Alice or for Bill in this game?
To make the game fair, how much should Alice pay to Bill instead of \$1?
Answer. It is $(1/2)$ probability to get an odd number in a single dice roll, and $(1/2)^3$ to get odd numbers three times. *On average*, in each turn Alice "wins"

$$\frac{1}{8}\cdot(+\$5)+\left(1-\frac{1}{8}\right)\cdot(-\$1)=\$\frac{5-7}{8}=\$(-0.25)<0$$

and she actually loses. To make the game fairer, she should pay less than \$1 if she loses, exactly $\$\frac{5}{7}$ to make it perfectly fair.

Homework

1. We take the standard card deck and draw one card. What is the probability that the card will be
(a) Queen of hearts
(b) Either a queen or a hearts card
(c) A red card
(d) A picture card (a jack, queen, king, ace)
(e) A picture card other than the queen of hearts

2. (a) What is the probability that when we toss a coin 4 times, there will be no heads?
(b) Alice and Bill are playing the following game. They toss a coin 4 times; if there are no heads, Alice wins, and Bill pays her \$10. Otherwise Alice loses and she pays \$1 to Bill.
Would you prefer to play for Alice or for Bill in this game?
(c) To make the game fair, how much should Alice pay to Bill instead of \$1?

3. (a) What is the probability that when we roll two dice, at least one will be a 6?
(b) Alice and Bill are playing the following game. They roll two dice; if at least one is a 6, Alice wins, and Bill pays her \$5. Otherwise Alice loses and she pays \$1 to Bill.
Would you prefer to play for Alice or for Bill in this game?
(c) To make the game fair, how much should Alice pay to Bill instead of \$1?

4. In a group of 100 students, 28 speak Spanish, 30 speak German, 42 speak French; 8 students speak Spanish and German, 10 speak Spanish and French, 5 speak German and French and 3 students speak all 3 languages. How many students do not speak any one of the three languages?
[Note: when it says that 28 students speak Spanish, this includes the 8 who speak Spanish and German; similarly for all other combinations.]

5. Supposing that there are equal chances of a boy or a girl being born, what is the probability that the first five babies born next Saturday morning at the St. Charles Hospital will be girls? That at least one of them five will be a girl?

6. (a) What is the probability that if we roll 2 dice, the sum will be at most 7?
(b) Alice and Bill are playing the following game. They roll 2 dice; if the sum is at most 7, Alice wins, and Bill pays her \$1. Otherwise Alice loses and she pays \$1 to Bill.
Would you prefer to play for Alice or for Bill in this game?
(c) To make the game fair, how much should Alice pay to Bill instead of \$1?

7. (a) What is the probability that if we roll 3 dice, all the numbers will be different?
(b) Alice and Bill are playing the following game. They roll 3 dice; if all numbers are different, Alice wins, and Bill pays her \$2. Otherwise Alice loses and she pays \$3 to Bill. //Would you prefer to play for Alice or for Bill in this game?
(c) To make the game fair, how much should Alice pay to Bill instead of \$3?