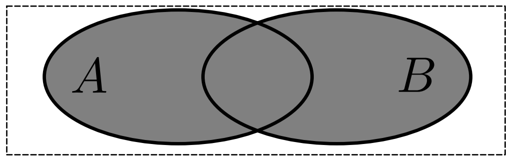
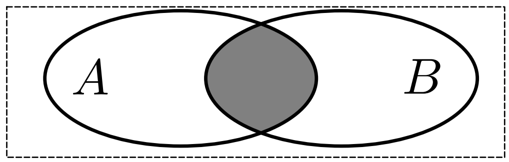
MATH 6 [2025 NOV 2]
HANDOUT 7: SETS II. INTERVALS AND COUNTING

SET OPERATIONS

There are several operations that can be used to get new sets out of the old ones:

- $A \cup B$: *union* of A and B. It consists of all elements which are in either A or B (or both):
$$A \cup B = \{x \mid x \in A \text{ OR } x \in B\}.$$


- $A \cap B$: *intersection* of A and B. It consists of all elements which are in both A and B:
$$A \cap B = \{x \mid x \in A \text{ AND } x \in B\}.$$


- $\overline{A}$: *complement* of A, i.e. the set of all elements which are not in A: $\overline{A} = \{x \mid x \notin A\}$.

INTERVALS

The following notations are used when we talk about intervals on the number line. Intervals may have end points included or excluded: [and] represent that the end point is included, while (and) indicate that the end point is excluded.

$[a, b] = \{x \mid a \le x \le b\}$ is the interval from a to b (including endpoints),
$(a, b) = \{x \mid a < x < b\}$ is the interval from a to b (**not** including endpoints),
$[a, \infty) = \{x \mid a \le x\}$ is the half-line from a to infinity (including a),
$(a, \infty) = \{x \mid a < x\}$ is the half-line from a to infinity (**not** including a)

COUNTING

We denote by $|A|$ the number of elements in a set A (if this set is finite). For example, if $A = \{a, b, c, \dots, z\}$ is the set of all letters of English alphabet, then $|A| = 26$.

If we have two sets that do not intersect, then $|A \cup B| = |A| + |B|$: if there are 13 girls and 15 boys in the class, then the total is 28.

If the sets do intersect, the rule is more complicated(see problem 8 below):
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

There is a special symbol for an empty set: a set containing no elements is $\varnothing$, so that $|\varnothing| = 0$.

PRODUCT RULE

If we need to choose a pair of values, and there are a ways to choose the first value and b ways to choose the second, then there are ab ways to choose the pair.

For example, a position on a chessboard is described by a pair like a4; there are 8 possible choices for the letter, and 8 possible choices for the digit, so there are $8 \times 8 = 64$ possible positions.

It works similar for triples, quadruples, ... For example, if we toss a coin, there are 2 possible outcomes, heads (H) or tails (T). If we toss a coin 4 times, the result can be written by a sequence of four letters, e.g. HTHH; since there are 2 possibilities for each of the letters, we get $2 \times 2 \times 2 \times 2 = 2^4 = 16$ possible sequences we can get.

HOMEWORK

1. Draw the following sets on the number line:

(a) Set of all numbers x satisfying $x \leq 2$ and $x \geq -5$;

(b) Set of all numbers x satisfying $x \leq 2$ or $x \geq -5$

(c) Set of all numbers x satisfying $x \leq -5$ or $x \geq 2$

2. For each of the sets below, draw it on the number line and then describe its complement:
(a) $[0, 2]$ (b) $(-\infty, 1] \cup [3, \infty)$ (c) $(0, 5) \cup (2, \infty)$

3. Long ago, in some town a phone number consisted of a letter followed by 3 digits (e.g. K651). How many possible phone numbers could there be in that town?[Note: digits could be zero, so a number like X000 was allowed.]

4. If we roll 3 dice (one red, the other white, and the third one, black), how many combinations are possible? How many combinations in which the sum of values is exactly 4?

5. Let

- A=set of all people who know French
- B=set of all people who know German
- C=set of all people who know Russian

Use the set notation to denote the following sets:

(a) Set of people who don't know Russian but know at least French or German.

(b) Set of people who know French and Russian but not German.

(c) Set of people who know all three languages.

(d) Set of people who don't know any of these languages.

(e) Set of people who speak at least two languages.

6. (a) Using Venn diagrams, explain why $\overline{A \cap B} = \overline{A} \cup \overline{B}$. Does it remind you of one of the logic laws we had discussed before?

(b) Do the same for formula $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. In this problem, we denote by $|A|$ the number of elements in a finite set A.

(a) Show that for two sets A, B, we have $|A \cup B| = |A| + |B| - |A \cap B|$. *Hint: try using Venn diagrams.*

*(b) Can you come up with a similar rule for three sets? That is, write a formula for $|A \cup B \cup C|$ which uses $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and also $|A \cap B \cap C|$ (the number of elements contained in sets A,B,C at the same time)?

8. In a class of 33 students, 12 are girls, 10 play soccer, and 10 play chess. Moreover, it is known that 6 of the soccer players are girls, that 2 of the chess players also play soccer, and that there is exactly one girl who plays both chess and soccer. Finally, 4 girls play neither soccer nor chess. Can you figure out how many boys play soccer? chess? both? neither?