MATH 6: HANDOUT 20
GEOMETRIC SEQUENCES

A sequence of numbers is a **geometric sequence** or **geometric progression** if if the next number in the sequence is the current number times a fixed constant called the **common ratio** or q.

Example: The sequence $6, 12, 24, 48, \dots$ is a geometric sequence because the next number is obtained from the previous by multiplication by $q = 2$.

We can also find the n-th term if we know the 1st term and q.

Example: What is a_{10} in the example above?

$$\begin{aligned} a_1 &= 6 \\ a_2 &= a_1 q = 6 \cdot 2 = 12 \\ a_3 &= a_2 q = (a_1 q) q = a_1 q^2 = 6 \cdot 2^2 = 24 \end{aligned}$$

The pattern is:

$$\begin{aligned} a_n &= a_1 q^{n-1} \\ a_{10} &= a_1 q^9 = 6 \cdot 2^9 = 6 \cdot 512 = 3072 \end{aligned}$$

Properties of a Geometric Sequence. Any term is the **geometric mean** of its neighbors:

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}$$

Proof: $a_n = a_{n-1} q$ so $a_n = a_{n+1}/q$ Multiplying these two equalities gives us:

$$a_n^2 = a_{n-1} \cdot a_{n+1}$$

from where we can get what we need.

Sum of a Geometric Sequence. Let's try to sum $1 + 2 + 4 + \cdots + 64$. For purposes of working with this sum, let it be called S, i.e. $S = 1 + 2 + 4 + \cdots + 64$. Then I can notice that $2S = 2 + 4 + 8 + \cdots + 128$; subtract the original sum to get $2S - S = 128 - 1$ (everything else cancels out). Thus $S = 127$. What did we do here? We multiplied by 2, which lined up the terms of the sequence to the next term over. In the geometric sequence 1, 2, ..., 64, the common ratio is $q = 2$.

Let's do this in general. Let $a_1, \dots, a_n$ be a geometric sequence with common ratio q.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1 - q^n)}{1 - q}$$

Proof: To prove this, we write the sum and multiply it by q:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= qa_1 + qa_2 + \cdots + qa_n \end{aligned}$$

Now notice that $qa_1 = a_2, \dots qa_2 = a_3, \dots, qa_n = a_{n+1}$, etc, so we have:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= a_2 + a_3 + \cdots + a_{n+1} \end{aligned}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$\begin{aligned} S_n - qS_n &= (a_1 - a_{n+1}), \text{ or} \\ S_n(1 - q) &= (a_1 - a_1 q^n) \\ S_n(1 - q) &= a_1(1 - q^n) \end{aligned}$$

from which we get the formula above.

Infinite Sum. If $0 < q < 1$, then the sum of the geometric progression is approaching some numbers, which we can call a **sum of an infinite geometric progression**, or just an **infinite sum**.

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The formula for the infinite sum is the following:

$$S = \frac{a_1}{1 - q}$$